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A Study of Mathematical Ability as Related to Reasoning and Use of Symbols.

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A STUDY OF MATHEMATICAL ABILITY
AS RELATED TO REASONING
AND USE OF SYMBOLS

by

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LIFE

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CHAPTER I

INTRODUCTION

With the growing demand for greater mathematical competence and the current emphasis on revision of mathematics programs in the schools it is becoming increasingly important to learn more about the psychological bases of mathematical thought and the principles that govern the learning of mathematics.

Mathematical thinking does not appear to be essentially different from any other kind of thinking but it does have a characteristic quality of its own. As Rosenbaum[1] points out, research on the foundations of mathematics has shown that all branches of mathematics can be reduced to purely abstract terms, with common properties. Algebra and geometry are both concerned with sets--algebra with sets of numbers, geometry with sets of points--and it is possible to deal with either in the same manner, using the same operations. It is not unreasonable, therefore, to expect to arrive at some basic parameters by means of the factorial techniques. According to Hadamard, "There is every reason to think that the mathematical faculty must be at least as complex as has been found for the faculty of language."[2]

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[1]E. P. Rosenbaum, "The Teaching of Elementary Mathematics," Scientific American (May 1958), 64-73.

[2]Jacques Hadamard, The Psychology of Invention in the Mathematical Field (New York, 1945), p. 6.

The purpose of this study is to contribute to the better understanding of the psychological nature of the numerical and reasoning factors that seem to enter into mathematical ability. Emphasis will be placed on the identification, by means of specially constructed tests, of the processes involved in mathematical thinking rather than on the identification of skills necessary for success in the study of mathematics at various grade levels.

No attempt is made here to define mathematical ability except to regard mathematics itself as a way of thinking--a science of reasoning.

Survey of Literature

Mathematical Ability in General

One of the earliest studies of mathematical ability was that of Collar.[3] This study was confined to arithmetic. Collar assumed that arithmetic had a trifold aspect: (1) computation, the mechanical aspect, (2) rules, the knowledge aspect, and (3) problems, the intelligence aspect. He administered a series of carefully planned tests in computation and what he called the higher arithmetical operations, namely, problems, rules, and mental arithmetic. Using Spearman's tetrad criterion he showed that the arithmetical abilities involved in those tests had much in common over and above *g*, the general intellective factor. This was taken as evidence of an arithmetical group factor. However, when he investigated the question as to whether this specific arithmetical ability might not fall into two subdivisions, a lower one for mere computation, and a higher one for the problems, rules, and mental arithmetic, he found that after the influence

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[3]Daniel Collar, "A Statistical Survey of Arithmetical Ability," *British Journal of Psychology*, XI (October 1920), 135-158.

of *g* had been eliminated, the inter-correlations of the arithmetical abilities were traceable to only a single factor. He concluded, therefore, that there was only a single specific mental factor operative in all kinds of arithmetical work and that the higher operations were different only in that they seemed to require more of *g*.

Spearman investigated another phase of the problem. It had been found that one group factor seemed to extend throughout geometry and another throughout arithmetic. Would these constitute a special ability for mathematics in general? For evidence on that point he re-analyzed a study of Rogers[4] and was led to conclude: "There appears, then, no real basis for the common opinion which would take arithmetic and geometry to furnish one single special ability. Their union as 'mathematics' seems, rather, to be merely one of practical convenience."[5]

A similar conclusion was reached by Fouracre.[6] On eliminating the effect of *g* he found only insignificant correlations between arithmetic and geometry, and between algebra and geometry.

In his latest book, *Human Ability*, published posthumously, Spearman still maintains, though somewhat reluctantly, the same view. He writes: "Particularly intriguing is the result obtained that, save for mere computation, arithmetical ability has nothing in common save only G. How is this conclusion ever going to make its peace with the generally accepted observation that many people of high

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[4]Agnes Low Rogers, *Experimental Tests of Mathematical Ability and Their Prognostic Value* (New York, 1918).

[5]Charles Spearman, *The Abilities of Man* (New York, 1927), p. 232.

[6]L. Fouracre, "Psychological Tests of Mathematical Ability," *Forum of Education*, IV (1926), cited by Spearman, Human Ability, p. 142.

general intelligence have, nevertheless, low ability for mathematics?"[7]

Oldham investigated the same problem. She proposed to find, by means of a statistical analysis, if there was a group factor for the three branches of school mathematics, arithmetic, algebra, and geometry, and how each branch correlated with intelligence. She found separate group factors for arithmetic, for algebra, and for geometry, but no large group factor for the three branches taken together, nor for any two of them when taken in pairs. The correlations of the three branches with intelligence were low. She observed: "the important factor dominating the correlations of mathematical abilities with intelligence is the extraneous factor of interest."[8] She adds: "The factor of interest enters very largely into all work in mathematics in schools."[9]

A very recent study by Wrigley led, among others, to the following conclusions:

> There is a close connection between mathematical and general ability; high intelligence is the most important single factor for success in mathematics.
>
> In addition, there exists a clearly identifiable mathematical group factor. The different branches of mathematics are linked together more closely than they would be if a general factor only were in operation.
>
> Performance in geometry is connected with spatial ability as measured by the spatial factor.

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[7]C. Spearman and Wynn Jones, Human Ability (London, 1951), p. 145.

[8]Hilda Oldham, "A Psychological Study of Mathematical Ability With Special Reference to School Mathematics," British Journal of Educational Psychology, VII (November 1937), 280.

[9]Ibid., 281.

> Performance in arithmetic (especially mechanical arithmetic), and to a lesser extent performance in algebra, is in part, dependent upon numerical ability as measured by a number factor.[10]

Blackwell[11] attempted to compare the factors involved in the mathematical ability of boys and girls in the range 13½ to 15 years. She found: (1) a central intellective factor of a reasoning nature resembling Spearman's g and common to both girls and boys, (2) a factor involving imagery and manipulation of spatial and verbal data also common to both sexes, (3) a verbality factor for girls and a verbal reasoning factor for boys, and (4) a tentatively interpreted factor of precision and exactness for girls.

Barakat conducted a study of mathematical abilities of boys and girls of a similar age group--13½ to 14½. Test intercorrelations for the two sexes were calculated separatelybut there were no major differences regarding the interpretation of factors. The intercorrelation matrices were analyzed (1) by the bipolar methods of simple summation and subdivided factors, and (2) by the method of group factors, both "non-overlapping" and rotated. The bipolar analysis suggested four dominant factors which Barakat interpreted as follows:

> (1) a general factor, roughly identifiable with general intelligence and common to all tests, (2) a group factor for mathematical ability common onlyto the tests involving numerical or mathematical processes, (3) a verbal factor, and (4) a visuo-spatial factor.
>
> There was strong evidence that the mathematical factor may itself be subdivided into narrower factors of a simpler type, notably a subfactor for mechanical arithmetic (closely related to memory) and another

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[10]Jack Wrigley, "The Factorial Nature of Ability in Elementary Mathematics," British Journal of Educational Psychology, XXVIII (February 1958), 61-78.

[11]A. M. Blackwell, "A Comparative Investigation Into the Factors Involved in Mathematical Ability of Boys and Girls," British Journal of Educational Psychology, X (June 1940), 143-153; (November 1940), 212-222.

for mathematical work (closely related to the manipulation of schemes and relations); and there seemed to be a cross division distinguishing geometry (which appeared to possess a high saturation with the spatial factor, probably in the form of visual imagery) from problem arithmetic and algebra (both of which seemed to depend on facility for dealing with formal variables).[12]

The studies of Oldham, Blackwell, and Barakat had one shortcoming in common. Each used a small number of tests. Oldham did not mention the exact number but the implication was that there were only a few. Blackwell administered fourteen but later grouped the geometry and the spatial tests into single tests so that only ten variables remained for the analysis. Barakat used thirteen tests.

A much more satisfactory study on that count was that of Weiss[13] who assembled a battery of thirty tests designed to measure aptitude and achievement in mathematics at the pre-college level. A total of thirty-two variables was analyzed. These included grades in algebra and geometry, tests in algebra and geometry, and reference tests for factors identified in earlier studies and expected to be found in algebra and geometry. Thurstone's multiple group method was used to obtain a factor matrix of ten factors. When the orthogonal axes were rotated to oblique simple structure nine factors were interpreted while the tenth was considered a residual. The factors were: (F) a vocabulary factor, (C) number facility, (H) spatial visualization, (A) geometry knowledge, (E) geometry grade, (B) skill in algebraic manipulation, (G) explicit reasoning, (D) implicit reasoning, (J) specific reasoning, and (K) the residual factor.

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[12]M. K. Barakat, "A Factorial Study of Mathematical Ability," British Journal of Statistical Psychology, IV (November 1951), 156.

[13]Eleanor S. Weiss, "A Factor Analysis of Mathematical Ability," Unpublished Doctoral Dissertation. Harvard University, Cambridge, 1955.

A second-order centroid analysis of the correlations between the nine primary factors yielded a strong general factor interpreted as a "schooling factor" underlying factors A, B, C, E, F, and G, all of which could be directly related to school tasks. Factors D, H, and J representing abilities that presumably receive little or no explicit training in school had loadings of -.11, .31, and -.36 respectively.

Kline[14] conducted two factor analyses of intermediate algebra. Thirty-eight tests were used with two successive classes. Of these tests, eighteen were tests of algebra and twenty were tests for reference variables aimed at isolating some of the factors found in earlier analyses. Separate tables of intercorrelations were computed for each of the two years and each was factor analyzed by the Thurstone complete centroid method. Both analyses supplied twelve factors. Three factors from each study were eliminated from the tests for congruence of factors in the two studies. Of the remaining nine factors only five were considered congruent to the two studies. These were: Factor A--Verbal Comprehension, Factor B--Deductive Reasoning, Factor C--Algebraic manipulative Skill, Factor D--Number Ability, and Factor E--Adaptability to a New Task. Only the first four were in evidence in the tests of algebra.

In a very comprehensive series of factorial and experimental studies Werdelin[15] attempted to clarify the factorial structure of mathematical ability particularly as related to problem-solving in school mathematics. Thirty-six tests were administered to more than 250 boys in the age range from 13+ to 16+. Some were later excluded and the final sample consisted of 217 boys. The battery

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[14]William E. Kline, A Synthesis of Two Factor Analyses of Intermediate Algebra, Educational Testing Service Technical Report (Princeton, 1956).

[15]Ingvar Werdelin, The Mathematical Ability (Lund, 1958).

included verbal, numerical, and spatial-perceptual tests. One of the main concerns in the construction of the mathematical tests was that they measure an ability considered to be of fundamental importance to school mathematics--understanding of a rule, translation of verbal statements into algebraic symbolism, etc. Each test covered a specific aspect of the subject matter of arithmetic, algebra, or geometry. The problems were similar to those used in school examinations. Six centroid factors were extracted. The rotations were performed by means of Thurstone's method of extended vectors. Five factors were interpreted: N, the numerical factor, V, the verbal factor, S, the visual factor, D, a deductive factor, and R, a general mathematical reasoning factor. A second-order general factor was closely related to the first-order factors D and R.

The Number Factor

In discussing the factors isolated in his classic studies of intelligence Thurstone explained that the number factor can be expected in any test in which simple arithmetical work is done. "The best tests for the number factor are the simple numerical tasks."[16] He cautioned against expecting the number factor in a test simply because it contains numbers. For example, he thought that it probably would not show up in a cancellation test where the subject is required to cancel out certain numbers, such as all the three's or five's on a given page. But if he is required to check every number that is larger than or smaller than the adjacent numbers, then the factor may be expected. He pointed out that the psychological nature of the number factor was not so clear as that of the other

[16] L. L. Thurstone and T. G. Thurstone, Factorial Studies of Intelligence (Chicago, 1941), p. 5.

primaries and that perhaps "number" as such was not an adequate description. He suggested that the factor N "may be more basic and general than number"[17] and that number tests were merely good examples of it.

Coombs undertook an investigation of the number factor mainly under the hypothesis that the number factor represents the agility with which an individual can manipulate a symbolic system according to a specified set of rules with two restrictions: that the symbolic system be familiar and that the set of rules be highly practiced.[18] A secondary purpose of his investigation was to obtain a clearer identification of the perceptual speed factor and to determine its relation to number ability. A battery of thirty-four tests was analyzed by Thurstone's centroid method. The results led to the following conclusions:

1. The number factor is most clearly identified by very simple number tests.

2. The perceptual speed factor is most clearly identified by cancellation tests, and the more scanning that is involved and the less cancelling the better the test is a measure of perceptual speed.

3. A test involving manipulation of a symbolic system is a better measure of number ability the more familiar the system.

4. A test involving operations according to a set of rules becomes a better measure of number ability with practice.

5. The data are in disagreement with the hypothesis that number ability is in the nature of a serial response.

6. The results are in agreement with the hypothesis that number ability is characterized by a facility in manipulating a symbolic system according to a specified set of rules.[19]

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[17]L. L. Thurstone, Primary Mental Abilities (Chicago, 1938), p. 83.

[18]Clyde H. Coombs, "A Factorial Study of Number Ability," Psychometrika, VI (June 1941), 161-189.

[19]Ibid., p. 188.

The non-numerical tests, especially devised to test the various hypotheses about the number factor, had vanishing loadings on the number factor. The above interpretation of the results was based chiefly on the analysis of test intercorrelations and the correlations of the tests with the number primary.

Werdelin proposed a new theory of the numerical factor. Supporting the view of Thurstone, that the number factor may be more general and broader than number, he writes: "It seems...as if it were illogical to presume a special ability for a cultural product like number; it seems more reasonable to assume that number is only an example of tasks that might be covered by the ability."[20] He agrees with Coombs that the numerical factor represents an agility in handling well-established associations but differs on the restrictions. "...the ability is characterized by the facility with which a person is handling a well-practiced (but not necessarily familiar) symbolic system according to a set of well-practiced rules, with one restriction: the process that has been practiced must originally involve some form of reasoning. The more difficult and comprehensive the rules, the more practice is needed before the tests will have a numerical loading."[21]

Werdelin's functional interpretation of the numerical factor is "the hypothesis that the ability behind the factor is characterized by the ability to automatize, by means of learning, the application of rules and principles, according to which tasks are solved which were originally solved by means of reasoning."[22] From this definition four sub-hypotheses were deduced:

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[20]Werdelin, p. 175.

[21]*Ibid.*, 175.

[22]*Ibid.*, 177.

1. A test involving processes, which have been reported to be automatizable, must show successively higher correlations with the numerical tests the more the process is practiced.

2. A test which involves a process that is more easily automatized than the process of another must approach the numerical tests more rapidly than the other.

3. A test, which involves a process that is not successively more automatized by means of practice must not show successively higher correlations with the numerical tasks.

4. A test which involves a process, which has first been automatized, but the automatization of which has been broken, must then show low correlations with the numerical tests.[23]

A battery of sixteen tests was prepared to verify these hypotheses experimentally. From a correlational analysis it was found that the results for a number of the tests supported the theories. An introspective study of the automatization of reasoning, made by means of a questionnaire regarding the degrees of automatization of both numerical and non-numerical tasks in the tests, also confirmed the hypothesis. A subsequent factorial analysis of a larger battery of thirty-seven tests[24] gave results which were again interpreted favorably.

Mathematical ability should not be confused with a talent for numerical calculation. Known cases of so-called "lightning calculators" and similar arithmetical prodigies indicates that a high level of numerical aptitude may occur in individuals of average or even inferior intelligence. Often the calculation seems to be elaborated unconsciously, without willful effort. Hadamard says that "Such a talent is, in reality, distinct from mathematical ability. Very few known mathematicians are said to have possessed it: one knows the case of Gauss

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[23] Ibid., 280.

[24] This larger battery included the sixteen tests used in the correlational analysis.

and Ampere and also in the seventeenth century, Wallis. Poincare confesses that he is a rather poor numerical calculator, and so am I."[25] The late John von Neumann used to astound other mathematicians with the speed with which he solved long and complicated problems in his head.

Reasoning

Several reasoning factors have been identified. Thurstone's Primary Mental Abilities study brought out an induction, a deduction, and a restrictive reasoning factor.[26] The inductive factor was characterized by tests such as Number Series where it was necessary to find a rule or principle for each item of the tests. The deductive factor was not very clear but the tests with significant loadings on it seemed to require the subject to find a rule and then to apply it to the given examples. The common characteristic of tests on the third reasoning factor was the solution of a task that required some form of restriction. The test with the highest loading on this factor was Arithmetical Reasoning.

The Army Air Forces Aviation (AAF) Psychology Program Report No. 5[27] describes several analyses that resulted in tentative interpretations of three reasoning factors designated simply as Reasoning I - a general reasoning factor best identified by arithmetic-reasoning tests, Reasoning II - possibly an inductive reasoning factor, and Reasoning III - a classification factor.

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[25]Hadamard, p. 58.

[26]Thurstone, Primary Mental Abilities, pp. 86-88.

[27]J. P. Guilford (ed) Printed Classification Tests, Army Air Forces Aviation Psychology Research Program Reports, Report No. 5 (Washington, 1947).

Thurstone's restrictive reasoning factor and the AAF general reasoning factor seemed to have much in common. Mathematical tests had high loading on both factors. Tests of arithmetical reasoning, word-problem type tests, were characteristic of both factors. Zimmerman[28] conceived the idea that if Thurstone's rotational process were continued a bit further the resulting reallocation of variance would yield factors that could be more closely identified with the AAF reasoning factors. Accordingly, he planned and performed further rotations of the Thurstone centroid axes with the expected results. The verbal tests that previously had high loadings on the restrictive reasoning factor now transferred large portions of their variance to the verbal factor while the mathematical tests made striking gains picking up variance from the induction factor. A classification factor was identified with the AAF Reasoning III factor and replaced Thurstone's induction factor. A new logical relations factor developed out of Thurstone's uninterpreted eleventh rotated factor.

In a study of factors related to intelligence Rimoldi[29] using mainly performance tests, found seven factors two of which are probably closely allied to the dynamics of reasoning. Factor C was described as an ability to overcome disturbing forces in the construction of a gestalt while factor B was interpreted as perception of relations in space necessary for the construction of a whole. Both factors seemed to transcend the test modalities.

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[28] Wayne S. Zimmerman, "A Revised Orthogonal Rotational Solution For Thurstone's Original Primary Mental Abilities Test Battery," Psychometrika, XVIII (March 1953), 77-93.

[29] H. J. A. Rimoldi, "Study of Some Factors Related to Intelligence," Psychometrika, XIII (March 1948), 27-46.

In another study,[30] using a battery of tests considered to be good measures of Spearman's g and of Thurstone's reasoning factors I, R, and D, Rimoldi found confirmation of the two factors mentioned above. Factors A and B isolated in the second study were identifiable with the factors C and B, respectively, of the previous study. In addition, reasoning factors described as "eduction of relations" (factor C), "eduction of correlates" (factor E), and "the eduction of likeness and its opposite" (factor D) were isolated. In the second-order factor α referred mainly to the relation of likeness, while factor β referred to the eduction of relations and the eduction of correlates. The relation of these factors to the principles of noegenesis[31] as described by Spearman was emphasized.

Reports from the Psychological Laboratory of the University of Southern California include several studies in the domain of reasoning. For the purpose of further research five factors were proposed: (1) General Reasoning--defining problems, (2) Logical Reasoning--application of logical rules, (3) Eduction of Correlates--completion of matching relations, (4) Eduction of Conceptual Relations, and (5) Eduction of Perceptual Relations.[32] The General Reasoning factor

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[30] H. J. A. Rimoldi, "The Central Intellective Factor," Psychometrika, XVI (March 1951), 75-101.

[31] As described by Spearman the principles of noegenesis refer to (1) the apprehension of one's own experience, (2) the eduction of relations, and (3) the eduction of correlates. The word was coined out of the Greek nous and genesis to designate the creation of knowledge in its two chief forms. Spearman considered it a unitary ability. See Abilities of Man, Chapter XI.

[32] J. P. Guilford et al., "A Factor-Analytic Study of Navy Reasoning Tests with Air Force Aircrew Classification Battery," Reports from the Psychological Laboratory, No. 6 (Los Angeles, 1952).

was exhaustively investigated.[33] It was finally concluded that the factor was best defined as an ability to structure problems. Whether the factor is so narrow as to include arithmetical problems only, or broad enough to include the forming of structures of comprehension apart from problems remains to be determined by further studies.

The Adkins and Lyerly[34] factor analysis of reasoning tests based on tests from the AAF batteries failed to confirm the general reasoning factor but did identify five reasoning factors. Three of them were thought to be inductive in character: Factor A' - Perception of Abstract Similarities, Factor B' - Hypothesis Verification, and Factor M' - Concept Formation. The other two were Factor H' - Deduction, and Factor F' - Flexibility of Perceptual Closure. Tests of a mathematical nature had significant loadings on only two of these factors. A number series test had a loading of .27 on Factor B'. An arithmetic test, a word-problem test in which the subject merely indicated the process to be used but did not solve the problem, had a loading of .29 on the Factor M'.

Two reasoning factors were identified in Werdelin's analyses of mathematical ability. The Deductive Factor D, found in all four studies, was characterized by tests of quantitative reasoning (syllogisms), number series and number analogies. Factor R, The General Mathematical Reasoning Factor was found in three of the studies. The factor was defined by mathematical tests but it was not considered a pure mathematical factor because the tests involved elements from many

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[33]J. P. Guilford, N.W. Kettner, and P. R. Christensen, "A Factor-Analytic Investigation of the Factor Called General Reasoning." Reports from the Psychological Laboratory, No. 14 (Los Angeles, 1955).

[34]Dorothy C. Adkins and Samuel B. Lyerly, Factor Analysis of Reasoning Tests (Chapel Hill, 1952).

sources: verbal, visual, numerical. The author described it as "the ability to combine the different elements, to group difficult problems, and to solve them by means of a process of reasoning."[35] This definition makes it highly probable that it is related to the General Reasoning factor identified in the AAF studies and sometimes referred to as a "trouble-shooting" ability.

Summing up the experimental evidence we may say:

(1) Past research has succeeded mainly in identifying the broad, general factors--visual, spatial, verbal, reasoning--that seem to enter into mathematical ability. There is a need to probe deeper to find basic underlying parameters.

(2) There is some agreement as to the role of the verbal, spatial, and visual factors. They appear to be complementary to mathematical activity and their relative importance is determined by the context of the tests.

(3) The nature of the number factor has not been clearly defined as yet.

(4) In studies of mathematical ability either a general reasoning factor or something closely related to it has generally been found in addition to variously defined inductive and deductive reasoning factors. The nature of mathematical reasoning has not been clearly defined.

(5) Mathematical tests used in the studies of mathematical ability have been almost exclusively in the nature of achievement tests measuring scholastic skills particularly routine mastery of computational skills.

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[35] Werdelin, p. 228.

CHAPTER II

THE PRELIMINARY STUDY

Even before a systematic review of pertinent bibliography had been completed, a preliminary analysis was planned in order to crystallize the problem and to help formulate the hypotheses to be advanced in the main study. This preliminary study was a factorial analysis of a sub-battery of fourteen of the fifty-seven psychological tests that were used by Thurstone.[1]

The tests selected were chosen either because they involved mathematical content or because they were tests of known factorial content and would help to identify the factors that would be extracted. A list of the tests is given in Table I.[2] Description of the tests together with sample items from each may be found in Thurstone's monograph.

A correlation matrix was set up using the published inter-test correlations of Thurstone. From this matrix of coefficients, given in Table II, eight factors were extracted by Thurstone's complete centroid method. After two successive factorizations the communalities were considered stable. The centroid factor matrix given in Table III, was rotated graphically into simple structure. Radial rotations and adjustments by the single plane method were used. The resulting

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[1]Thurstone, Primary Mental Abilities.

[2]Appendix I, p. 62.

oblique factor matrix is given in Table IV, and the final transformation matrix in Table V. The cosines of the angles between the reference vectors are given in Table VI and the correlations between the primaries in Table VII.

The Factors

In listing tests with significant factor loadings values of .20 and higher are included though the interpretation is based mainly on tests whose loadings are .30 or higher.

Factor A

Test		Loading
3	Pursuit	.60
4	Areas	.46
12	Patterns Analogies	.42
14	Hands	.31
1	Figure Classification	.23

All these tests are of a visual, spatial character. This factor is, therefore, identified with the Thurstone S, or space factor.

Factor B

Test		Loading
6	Multiplication	.59
9	Numerical Judgment	.50
7	Tabular Completion	.43
10	Arithmetical Reasoning	.43
5	Number Code	.36
8	Number Series	.33

Since the test with the highest loading is Multiplication, this factor is identified with Thurstone's N, or number factor.

The cluster of tests on factor B included all the tests that require any calculation at all and no others. All those tests with the exception of Multiplication have significant loadings on reasoning factors as well.

Factor C

Test		Loading
8	Number Series	.51
1	Figure Classification	.38
10	Arithmetical Reasoning	.24
7	Tabular Completion	.21

This factor does not correspond directly to any of the Thurstone factors. It appears to be a reasoning ability, particularly that of educing relations.

Factor D

Test		Loading
12	Pattern Analogies	.48
4	Areas	.43
9	Numerical Judgment	.43
10	Arithmetical Reasoning	.38

This factor is interpreted as the ability to educe correlates. Factors C and D taken together seem to account for Thurstone's I, inductive reasoning, but here they appear to be quite distinct with only test (10) Arithmetical Reasoning appearing on both. The correlation between the factors C and D is quite high (.79).

Factor E

Test		Loading
2	Identical Forms	.50
11	Code Words	.42
13	Syllogisms	.35
12	Pattern Analogies	.27

Factor E is difficult to interpret but it appears closest to Thurstone's factor P, perception.

Factor F

Test		Loading
5	Number Code	.53
14	Hands	.42
11	Code Words	.35
10	Arithmetical Reasoning	.25

Factor F may represent abstraction. It is somewhat similar to Thurstone's factor XI, which was not interpreted by him.

Factor G

Test		Loading
13	Syllogisms	.38
7	Tabular Completion	.36
4	Areas	.36

Factor G is difficult to interpret. It seems to be a reasoning factor of some kind, perhaps of a deductive nature.

Factor H

Test	Loading
11 Code Words	.29
13 Syllogisms	.20

Factor H is a residual factor.

Conclusions

Mathematical activity as shown in the tests of this small battery appears to be a complex combination of several distinct mental processes. The mathematical tests were found on five of the factors--number, abstraction, and three factors of a reasoning nature.

Hypotheses For The Main Study

An integration of the implications drawn from the findings of previous research, including the preliminary study, and from the writings of mathematicians, psychologists, and educators has led to the development of the following hypotheses:

1. Mathematical ability involves essentially:

   a) Ability or abilities to see or discover relations, realize their implications, and make inferences from them.

   b) Ability to educe correlates, to extract from given data facts not explicitly stated.

   c) A fluency in the manipulation of certain symbols; an ability to handle abstract qualities without concrete aids.

   d) Ability to analyze a situation, distinguish relevant from irrelevant data, and organize a sequence of steps leading to a solution. This ability may be considered a function of the abilities assumed in _a_, _b_, and _c_ above.

2. The number factor commonly associated with mathematical ability is not limited to purely computational aspects. Computational facility may or may not be an important element in mathematical ability.

3. Other factors enter into mathematical ability in varying degrees depending upon the context of the problem or the nature of the task.

4. An aesthetic sense of symmetry, order, and harmony is characteristic of many phases of mathematical activity.

5. To get at the basic, underlying factors that determine the dynamics of mathematical thinking the tests used in the study ought to be of such a nature as to depend little, if at all, on formal training in the particular subject branches but rather require the use of natural intuition and innate abilities of a fundamental kind.

It is particularly in connection with the last point that the main study was planned. The study does not intend to cover all the branches of mathematics, nor any branch in particular, but we have used tests built on some ideas that are at the basis of mathematical thinking, such as, limits, functions, equality, inequalities, rearrangement of concepts, etc., rather than on what schools call mathematics.

General Description of the Variables

A battery of thirty-six tests was assembled. Ten of the tests were standardized tests. All the other tests were constructed by the writer. Of these, twelve were patterned after tests used by other investigators in previous studies, and fourteen were original, new tests developed especially for this study.

The tests are described in Appendix II. Each test is illustrated by one or more sample items. Tests 1, 19, 20, 21, and 22 were reference tests for the number factor. Tests 2, 8, 10, 11, 12, 17, and 18 were reference tests for reasoning factors. Tests 4 and 5 were reference tests for the verbal factors and test 3 for the space factor. Tests 6 and 7 were included as external criteria of mathematical ability. The eighteen experimental tests that completed the battery were designed to explore hypotheses about the nature of mathematical ability. A list of the variables is given below.

<u>List of Variables</u>

<u>Standardized Tests</u>

1 PMA Number

2 PMA Reasoning

3 PMA Space

4 PMA Verbal-Meaning

5 PMA Word-Fluency

6 California Mathematics Tests (CMT), Reasoning

7 California Mathematics Tests (CMT), Fundamentals

8 Holzinger-Crowder Uni-Factor Tests, Mixed Series

9 Holzinger-Crowder Uni-Factor Tests, Figure Changes

10 Holzinger-Crowder Uni-Factor Tests, Teams

<u>Adapted Tests</u>

11 Number Series 1

12 Number Series 2

13 Statement Translation

14 Functional Relationship

15 Problem Analysis 1

16 Problem Analysis 2

17 Figure Grouping

18 Figure Matrix

19 Addition

20 Subtraction

21 Multiplication

22 Division

<u>New Tests</u>

23 Conditions 1

24 Conditions 2

25 Fluency with Mathematical Expressions

26 Quantitative Relationship

27 Numerical Inequalities

28 Algebraic Inequalities

29 General Expressions

30 Number Oddities

31 Number Relations

32 Number Fluency

33 Formulas and Figures

34 Mixed Operations

35 Missing Number

36 Missing Sign

CHAPTER III

THE MAIN STUDY

Sample of Subjects

There is some experimental evidence that a significant difference in performance on mathematical tests exists for the two sexes.[1] Boys and girls apparently use either different abilities or they use the same abilities differently when they do such tests. Practical considerations, however, made it necessary to confine the present study to girls only.

The full battery of thirty-six tests was administered to 160 eleventh-grade students in a private secondary school for girls. Complete data were obtained for 150 and these constituted the experimental group. The average I.Q. as measured by the California Test of Mental Maturity was 110 with a S.D. of 8.7. The frequency distribution of the I.Q.'s for the group is shown in Table VIII. The average grade placement on the California Mathematics Test, Advanced Form AA (1950 edition) was 12.9, at the 75th percentile. The 1957 norms applied to the raw scores on the 1950 edition by means of conversion tables supplied by the California Test Bureau gave a grade placement of 13.1, at the 80th percentile.[2]

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[1]Anne Anastasi and John P. Foley, _Differential Psychology_ (New York, 1949), pp. 657-660. F. J. Houlihan, "Secondary School Boys' and Girls' Achievement and Intelligence," _The Catholic Educational Review_, LI (May 1953), 289-299.

[2]The average raw score was 103.3, out of a possible 140, with a S.D. of 16.52.

The age distribution of the group is shown in Table IX. The average age was 16 years, 5 months.

The mathematical background for all the subjects was very similar. All had had a year of algebra. One hundred twenty-nine (86 per cent) had algebra in the ninth grade and plane geometry in the tenth. The others had general mathematics in the ninth and algebra in the tenth grade. In an informal questionnaire given after all the tests had been taken 129, that is, 86 per cent indicated that they liked mathematics and had enjoyed taking the tests. Even those who said that they did not like mathematics admitted that they had found the tests interesting.

Testing Procedures

For the standardized tests, the students used the self-scoring answer sheets provided by the publishers. The other twenty-six tests were so prepared that the answers could be written directly on the test sheet with a minimum of coding. All the tests were hand-scored.

Time limits as given in the respective manuals for the standardized tests were strictly observed. Tentative time limits were set for the remaining tests and all twenty-six were administered in preliminary from to two classes of high-school sophomores and a group of seniors, a total of 100 students. This pre-trial led to modification and clarification of the test directions, some minor changes in the presentation of test items and in the recording of answers, and a final determination of time limits. The tests were to be primarily power tests and the time limits were generous, allowing almost every student to attempt all the items. A list of the tests with time limits, number of items, and scoring formulas is given in Table X.

The tests were administered by the writer and by seven other members of the faculty during regular class periods. Since there was no room large enough to accomodate the whole experimental group comfortably, the tests were given in class groups of thirty to thirty-five students. To assure uniformity of testing procedures, detailed instructions for the administration of each test were provided. All the testing was completed within four weeks--the last two weeks of September and the first two weeks of October, 1958.

Statistical Procedures

Table XI lists the descriptive statistics for each test. The reliabilities are stepped-up, even-odd reliability coefficients. Even-odd reliability coefficients are underestimates of the reliability of power tests, particularly if the tests contain a small number of items. This seems to be true of the few tests with low reliabilities in this study. Though low reliability does tend to lower the factor loadings it presumably does not affect the factor structure.[3]

The raw scores were converted to standard scores and then scaled by means of graphic interpolation so that each raw score could be expressed as a positive, single-digit number. Pearson product-moment correlation coefficients were then computed with the aid of a desk calculator. The matrix of test intercorrelations is given in Table XII. The frequency distribution of the test intercorrelations is shown in Table XIII. The mean of the distribution is 0.32. The correlations are positive with only a few exceptions. The seven negative correlations are all with Test 25, Fluency with Mathematical Expressions.

[3]J. P. Guilford, Psychometric Methods (New York, 1954), p. 532.

CHAPTER IV

THE FACTORIAL ANALYSIS

Extraction of Factors

The thirty-six variable correlation matrix was factored by the complete centroid method by means of a high-speed digital computer, the Illiac. The first communality estimates for the tests were computed by Thurstone's centroid formula. Later, at each stage of the factoring the highest entry in the array was used as the new communality estimate. Twelve centroid factors were extracted. The projections of the test vectors on these arbitrary, orthogonal reference vectors are given in the centroid factor matrix in Table XIV.

Rotation to Simple Structure

Initial rotation of the centroid factor matrix to an oblique simple structure was made analytically by Oblimax, the program by which the Illiac is said to rotate automatically to simple structure. Thurstone[1] advises, however, that final acceptance of a simple structure should always depend upon the appearance of the graphs constructed by plotting the values in the columns of the rotated factor matrix.[2] In a recent comparative study of some analytic methods of

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[1]Thurstone, Multiple Factor Analysis (Chicago, 1947), p. 377.

[2]The columns are plotted against each other two at a time. For r factors there will then be $\frac{1}{2}r(r-1)$ such plots. In the present twelve-factor solution sixty-six plots are made for each rotation.

rotation Gocka[3] used the known "subjective-graphical" solutions of selected problems as the standard criterion.[4] In five of the seven problems studied, the analytic methods gave results that only approximated the subjective-graphical standard factors. In each of these cases he concluded that final graphical adjustments would have to be made to obtain a good simple structure.[5] In general, analytic procedure s gave satisfactory correspondence with the subjective-graphical standard in cases where the data represented a configuration with extremely well defined primary axes. Such cases are "optimal" for producing a simple structure.[6] On data representing less optimal conditions (as may happen in exploratory studies) the analytic methods were found to be inadequate. In the present study, the solution obtained by the analytic method was plotted. As the diagrams were examined it became evident that the structure was not compelling and that all the criteria for simple structure as given by Thurstone were not satisfied. It was necessary to make adjustments graphically. The radial and single-plane methods of rotation were used. Twenty additional rotations were taken. A complete set of the final plots is presented in Appendix IV. Most of the diagrams show a concentration of points at or near the origin and a definite grouping of several points along the two axes.

The projections of the tests on the oblique reference vectors are listed in Table XV. The final transformation matrix which relates the initial, arbitrary,

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[3]Edward F. Gocka, A Comparison of Some Analytic Methods of Rotation In Factor Analysis (Seattle, 1959).

[4]The standard criterion procedure consists of an element by element comparison of an analytically rotated factor matrix with the graphically rotated factor matrix for the same data.

[5]Gocka, Comparison of Analytic Methods, pp. 82-91.

[6]Ibid., pp. 107-108.

orthogonal solution to the oblique simple structure solution is given in Table XVI. The cosines of the oblique reference vectors are given in Table XVII.

Interpretation of the Factors

The rotated factor matrix of Table XV is summarized in Table XVIII. All loadings less than .20 have been omitted so as to provide a clearer picture of the nature of the simple structure. It can readily be seen that several of the tests are complex, having significant loadings on more than one factor. In an exploratory study of this type that may be expected.

The interpretation of each factor is based primarily on the tests with high loadings, that is, .30 or higher, but occasionally tests with lower loadings are used to give supplementary evidence.

Factor A

Test		Loading
22	Division	.43
19	Addition	.38
34	Mixed Operations	.37
1	PMA Number	.36
12	Number Series 2	.35
20	Subtraction	.34

All the tests on this factor deal with numbers. Division, Addition, and Subtraction have been classically considered as tests loaded in the number factor. PMA Number is also present here. It seems reasonable to think that this factor represents what has been known as the number factor described by Thurstone.[7]

[7] Thurstone, Primary Mental Abilities, p. 82.

Mixed Operations and Number Series both involve simple numerical work. In Mixed Operations the four fundamental operations are performed in a heterogeneous manner. In Number Series it is necessary to discover the number that "does not belong." The calculations involved are mainly simple addition and subtraction.

It may be noted that Multiplication has no loading on factor A and that all the tests on this factor, except for Addition, have loadings on other factors as well.

Factor B

Test		Loading
30	Number Oddities	.58
2	PMA Reasoning	.44
3	PMA Space	.39
18	Figure Matrix	.32
9	Figure Changes	.31
12	Number Series 2	.23
35	Missing Number	.23
36	Missing Sign	.21

The factor clearly transcends the material of the tests and is not limited to one particular type. Number Oddities, Number Series, Missing Number and Missing Sign deal with numbers. PMA Reasoning uses letters as do some of the items in Number Oddities. Figure Matrix, Figure Changes, and PMA Space involve forms and figures.

A study of the mental processes that are involved in doing these tests indicates that they seem to require the ability to handle several more or less conflicting Gestalts either simultaneously or in succession. In Number Oddities it is necessary to recognize a pattern within a pattern and as in PMA Reasoning discover the principle on which the pattern is built. In PMA Space and in Figure Changes the subject must keep in mind a given figure or relation while examining several alternates which must be compared with the given figure. A similar activity is required in Figure Matrix.

Factor B might be identified with Thurstone's factor E (Flexibility of Closure) which he described as an ability that facilitates the retention of a figure in a distracting field.[8] Flexibility of Closure factors identified in other studies[9] were defined by such tests as Figure Classification, Identical Forms, Gottschaldt Figures, and similar tests of a perceptual nature. In this study, PMA Space, Figure Matrix and possibly Figure Changes could come under that classification. All three tests have high loadings on factor B.

It appears that the Flexibility of Closure factor can be extended beyond the perceptual domain. Pemberton's study was thought to yield strong evidence that flexibility of closure is associated with analytical reasoning. In the present study Number Oddities and PMA Reasoning are series type tests where it is necessary to manipulate the parts separately and yet maintain a clear picture of the whole configuration. Both tests have high loadings on factor B. PMA Reasoning is similar to Number Series 1. It may be interesting to note that Number Series 1 has no significant loading on factor B while Number Series 2 has at least a

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[8] L. L. Thurstone, "A Factorial Study of Perception," Psychometric Monographs, No. 4, 1944.

[9] Dorothy C. Adkins and Samuel B. Lyerly, Factor Analysis of Reasoning Tests, (Chapel Hill, 1952). Wm. Botzum, "A Factorial Study of the Reasoning and Closure Factors," Psychometrika, XVI (Dec. 1951), 361-386. Carol Pemberton, "The Closure Factors Related to Other Cognitive Processes," Psychometrika, XVIII (Sept. 1952), 267-288.

small loading on the factor.

Solution of items in Missing Number and Missing Sign appears to be greatly facilitated by the ability to see the equation as a whole and move the signs and number about mentally until the equality is satisfied. In the easier items not much insight is needed. They can readily be solved by inspection. The more difficult items, however, require a synthetic activity combined with analytical processes.

Factor C

Test		Loading
17	Figure Grouping	.34
35	Missing Number	.29
27	Numerical Inequalities	.26
34	Mixed Operations	.23
24	Conditions 2	-.24

The interpretation of this factor can only be very tentative because of the small number of tests defining it and their low loadings.

Figure Grouping is a Classification test. The items are easy and probably require little more than perception of the common property that distinguishes three of the figures from a fourth one which is different.

The nature of the tests Missing Number, Numerical Inequalities, and Mixed Operations suggests an element of flexibility of operation. In each of these tests it is necessary to shift from one operation to another and handle almost simultaneously addition, subtraction, multiplication, and division. Missing Number has a small (.23) loading on factor B which was identified with flexibility of closure.

It is interesting to note that Conditions 2, a difficult reasoning test of an abstract nature, has a negative loading on factor C.

It is probably best to consider factor C a residual factor. Factor C has one of the highest second-order loadings.

Factor D

Test		Loading
24	Conditions 2	.42
10	Teams	.35
23	Conditions 1	.34
12	Number Series 2	.33
13	Statement Translation	.32
26	Quantitative Relationship	.31
25	Fluency with Math. Expressions	.30
29	General Expressions	.28
16	Problem Analysis 2	.26
31	Number Relations	.23
22	Division	.23
6	CMT Reasoning	.20

Conditions 1 and Conditions 2 were constructed as abstract, mathematical counterparts of Teams, a test of syllogistic reasoning. These three tests head the list on factor D. In Teams and in Conditions 1 the relation is stated and the subject must judge it true or false under the given premises. In Conditions 2 the relations are not given but must be educed by the subject.

Number Series 2 requires the discovery of the principle on which the series is built and finding the number that does not belong.

In Statement Translation and in Fluency with Mathematical Expressions it was necessary to express in mathematical symbols a relation given in verbal form.

Quantitative Relationship was designed as a test for the relation of likeness in a mathematical context. Two members of an expression are given and the subject must judge whether the first is equal to, greater than or less than the second.

General Expressions was considered by the students a difficult test. It required the generalization of a principle and the eduction of an expression defining the relationship.

Factor D seems to have some of the properties of formal logic--ordering, scaling, organizing.

In all the tests defining factor D it seems that two things were necessary for the successful completion of the task: the ability to manipulate simultaneously a number of relationships and the ability to use symbols, both of which suggest that a certain amount of flexibility is an important factor in mathematical reasoning processes. The criterion test CMT Reasoning has a small (.20) loading on this factor which could be significant in terms of the view expressed above, particularly if we observe that its companion criterion test CMT Fundamentals has a loading of .00 on the factor. In CMT Reasoning the various items in each of the four parts were very unlike. No fixed "set" could be maintained in working any of the parts. Each of the four sections in CMT Fundamentals, on the other hand, involved one specific type of operation only: addition, subtraction, multiplication, or division.

Factor E

Test		Loading
27	Numerical Inequalities	.36
12	Number Series	.32
24	Conditions 2	.30
32	Number Fluency	.28
31	Number Relations	.27
10	Teams	.25
11	Number Series 1	.20
25	Fluency with Math. Expressions	-.43

In Numerical Inequalities and in Conditions 2 the task is to state the relationship between the members of a given expression. The relation is essentially that of likeness: equal to, greater than, or less than.

In Number Series 2 it is necessary to discover the number that disturbs the sequence--a number that is unlike the others in the row.

Number Fluency calls for the eduction of numbers all of which must satisfy a given condition, hence a common relation of likeness.

If the factor represents the ability to manipulate quantitative relationships then the high negative loading of Fluency with Mathematical Expressions is not so strange. This test involves the handling of operational relations and a versatility in translating from verbal to mathematical symbols.

Another interpretation for the factor may be that it represents the ability to work under limiting, restrictive conditions. It may represent the handicap under which much mathematical activity must be performed--limitations and restrictions imposed by the hypotheses, definitions, and postulates more or less arbitrarily set up.

Factor F

Test		Loading
14	Functional Relationship	.48
6	CMT Reasoning	.42
15	Problem Analysis 1	.42
7	CMT Fundamentals	.40
28	Algebraic Inequalities	.33
13	Statement Translation	.23
20	Subtraction	.22
26	Quantitative Reasoning	.21

Factor F is clearly dependent on formal training in mathematics. Functional Relationship involved interpretation of direct and inverse variation. The criterion tests, CMT Reasoning and CMT Fundamentals were standard achievement tests in mathematics. The simple word problems in Problem Analysis 1 were typical of test items on current Arithmetic Reasoning tests. Algebraic Inequalities and Statement Translation as well as Quantitative Relationship required skills regularly developed in routine classroom work.

Factors defined by scholastic variables have been isolated in the first order by Comrey[10], Holzinger and Swineford[11], and Carroll[12], and in the second

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[10]A. L. Comrey, "A Factorial Study of Achievement in West Point Courses," Educational and Psychological Measurement, IX (Summer 1949), 193-209.

[11]K. J. Holzinger and F. Swineford, A Study in Factor Analysis: The Stability of a Bi-Factor Solution, Supplementary Educational Monographs, No. 48,

[12]J. B. Carroll, "The Factorial Representation of Mental Ability and Academic Achievement," Educational and Psychological Measurement, III (1943), 307-332.

order by Weiss.[13] In all these studies, however, tests and grades were used as variables. Whenever grades are used as variables extraneous, complex elements are introduced. The factors defined by a combination of test and grade variables have been described in terms of such personality characteristics as "ambition," "will to succeed," "will to learn."

The present factor F cannot be identified with those scholastic factors where grades are the chief determining variables.

Factor G

Test		Loading
28	Algebraic Inequalities	.53
33	Formulas and Figures	.46

Factor G is only a doublet and its interpretation is uncertain. It is probably a factor unique for this study, caused by the nature of the two tests. Either the tests were too difficult, or the tasks too unfamiliar. At any rate, it seems that an element of uncertainty permeated the performance of many subjects on these tests. At best, the factor might be described as a "guessing" factor. It is striking that the two tests so characterized should bring out a distinct factor with so much sharpness.

Factor H

Test		Loading
4	PMA Verbal-Meaning	.41
35	Mixed Operations (n)	.29
36	Mixed Operations (s)	.28

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[13] Weiss, An Analysis of Mathematical Ability

34	Mixed Operations 2	.27
29	General Expressions	.27
3	PMA Space	.22
12	Number Series 2	.21

Test 4, PMA Verbal-Meaning, is one of the standard reference tests for the verbal factor. The other tests on Factor H are all non-verbal. Their verbal content is confined to the instructions. The small but probably not insignificant loadings of these tests might be explained by the observation that many subjects found it easier to solve the problems by verbalizing the operations that were to be performed. The name Verbalization thus seems more appropriate than the more commonly used Verbal-Meaning or Verbal Factor. The attempt to minimize the verbal factor in the battery was apparently successful. The verbally presented tests, Statement Translation, Problem Analysis I and II, have loadings of only .03, .10, and .09 respectively on Factor H.

Factor I

Test		Loading
21	Multiplication	.43
10	Teams	.40
8	Mixed Series	.33
7	CMT Fundamentals	.22

All the tests on this factor deal with numbers except Teams which is a test of non-quantitative, syllogistic reasoning.

Multiplication has traditionally been known as one of the best reference tests for the number factor. Mixed Series obviously requires manipulation with numbers. CMT Fundamentals consists of four short achievement tests based on problems in which the fundamental operations are used with whole numbers, fractions, and decimals.

Thus it seems that this factor is possibly another number factor and perhaps represents the ability to perform simple numerical operations where these are not presented as mere number manipulation. The high loading of Teams on this factor is difficult to interpret.

Multiplication has an average correlation of .36 with the tests on factor A but its loading on factor A is only .09. Factor I correlates .65 with factor A which indicates a strong relationship between the factors. There are no overlapping tests. Thus the two number factors are quite distinct.

<u>Factor J</u>

Test		Loading
9	Figure Changes	.57
14	Functional Relationship	.34
8	Mixed Series	.33
2	PMA Reasoning	.30

Figure Changes, Mixed Series, and PMA Reasoning are tests of the type that Spearman considered to be good tests of "g". Such tests, Thurstone observed, seem to be inductive in character. In this battery all three are factorially complex.

Figure Changes and Functional Relationship are tests in which the relationship and part of the answer are given. The mental activity required to complete the item is essentially the eduction of a correlate. In Mixed Series and in PMA Reasoning the relationship is not given explicitly, but once it is discovered, the response may be the eduction of a correlate--a letter or number that would continue the series.

Factor K

Test		Loading
33	Formulas and Figures	.46
8	Mixed Series	.43
18	Figure Matrix	.32
36	Missing Sign	.26
34	Mixed Operations	.23

Factor K is difficult to interpret with confidence because all the tests that define it are factorially complex.

Mixed Series and Figure Matrix are inductive in character. In Formulas and Figures an algebraic expression is to be matched with a geometric figure--relationships are sought. The ability to educe abstract relations appears to be the main component of the mental processes that are probably involved in the solution of the tests represented on this factor.

Factor L

Test		Loading
32	Number Fluency	.43
5	PMA Word-Fluency	.32
1	PMA Number	.24
12	Number Series 2	-.36

Factor L is obviously a fluency factor. Both tests, Number Fluency and PMA Word-Fluency, required the subject to produce quickly responses under a simple restricting condition. In Word-Fluency the words had to begin with a given letter; in Number Fluency the numbers had to satisfy a given condition: even, odd, multiples of 3, etc.

The PMA Word-Fluency test is sometimes considered to be a test of some degree of creativity or ideational fluency. PMA Number also has a small loading on this factor. It seems that the problems may have been solved in a more or less automatic way because of familiarity with the material.

The test, Number Fluency, was developed for the purpose of discovering whether there would be any relation between word-fluency and number fluency. Apparently there is and the link may well be a certain kind of reaction-time factor, the speed and ease with which one characteristically reacts to a familiar stimulus. Thus fluency may be a personality characteristic.

Two other tests, Number Relations and Fluency with Mathematical Expressions, were designed to study the relation between word fluency and number fluency. The restrictive conditions, however, being more rigid, these tests have vanishing loadings on factor L.

Second-Order Analysis

Correlations Between the Primary Factors

When factorial analysis is extended to the second order where an attempt is made to account for the correlations among the primaries, then more fundamental psychological variables are expected to emerge. Since the purpose of this study was primarily the determination of some basic factors underlying mathematical ability, a thorough second-order analysis was performed.

The correlations between the twelve primary factors are shown in Table XIX. Among the high correlations we note that the highest is the correlation of .66 between the two number factors, A and I. Both factors also have high correlations with factor H, Verbalization.

Factors B, C, and D form one highly correlated cluster while factors F, H, and L form another. Factor L actually overlaps both clusters.

Factors E and K have moderate correlation with each other but seem to be independent of the rest of the factors. Factor G also appears to be independent of the other factors except for its relatively high negative correlation with factor K. Factor J is practically orthogonal to all the other factors.

The Second-Order Factors

The matrix of correlations among the primaries (Table XIX) was factored by the multiple-group method. Communalities were estimated by means of Thurstone's centroid formula and adjusted after each of the three successive factorizations. Four factors were extracted. The orthogonal second-order factor matrix is given in Table XX. The loadings of the tests on the unrotated second-order factors are listed in Table XXIV.

Rotation of the orthogonal second-order factors to an oblique simple structure was performed graphically and also by the analytical method developed by Thurstone.[14] The first three second-order factors A', B', and C' were determined almost identically by the two methods; the fourth factors differed. The simple structure obtained by graphical rotation was better than that achieved by the analytical method. The second-order analysis is therefore based on the results of the graphical rotations. The rotated second-order oblique factor matrix is given in Table XXI, the final transformation matrix in Table XXII, and the correlations between the second-order factors in Table XXIII. Loadings of the tests

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[14] L. L. Thurstone, "Analytical Method For Simple Structure," Reports from the Psychometric Laboratory, University of North Carolina, No. 16 (July 1953).

in the second-order rotated factors are listed in Table XXV.

Factor A' is almost orthogonal to the other second-order factors. Factor C' and D' show small negative correlation. Factor B' has positive correlation with C' and negative with D'.

The "g" loadings for the tests were computed by means of Thurstone's formula (51).[15] They are listed in Table XXVI.

The loadings of the tests on the first unrotated second-order factor were plotted against their "g" loadings. The result is shown in Figure 1.

Allowing for rounding errors the closeness of the two sets of values is such as to indicate that the first unrotated second-order factor is a pretty good approximation of the general factor that would have been extracted if Spearman methods of analysis had been used. The result is not as close as that reported by Rimoldi.[16] This may be partly due to the difference in the composition of the two test batteries. In the present study the tests were of a more homogeneous nature; of the thirty-six tests in the battery, twenty-eight were mathematical.

Interpretation of Second-Order Factors

The rotated second-order factor matrix has been summarized in Table XXVII. Primary factor J is not listed because it had only a small loading in each of the second-order factors.

Factor A' is defined by the four primaries C, D, B, and L. It is thus quite general but has neither a verbal nor a numerical factor. The characteristic

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[15] Thurstone, Multiple Factor Analysis, p. 276.

[16] H. J. A. Rimoldi, Central Intellective Factor, p. 94.

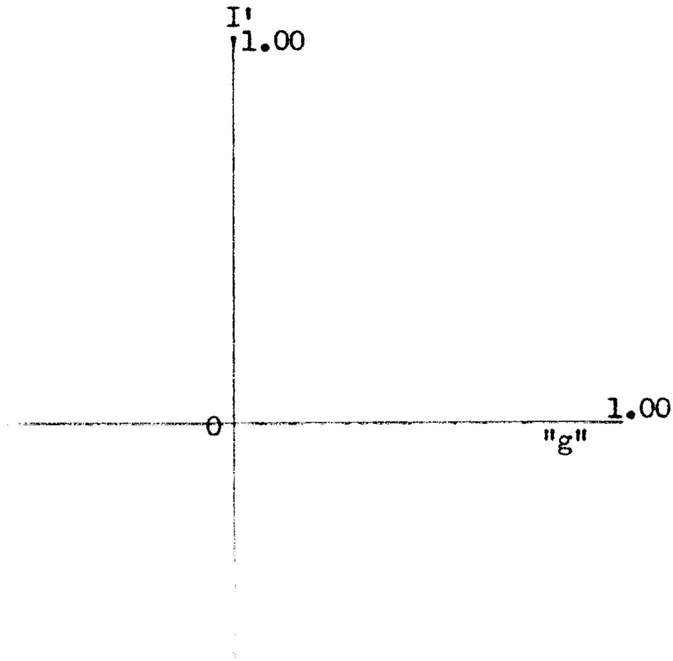

FIGURE 1

PLOT OF "g" LOADINGS AGAINST LOADINGS IN THE FIRST UNROTATED SECOND-ORDER FACTOR element appears to be flexibility in handling complex perceptual relations and abstract mathematical symbols.

Factor B', defined by the two number primaries A and I, is clearly a number factor. It may be significant that the primary D, defined almost exclusively by tests of a mathematical nature, has a rather high negative loading on this second-order number factor.

Factor C' probably represents abstraction. Both primary factors K and E which define it require the ability to educe and manipulate relations of all kinds. If the interpretation of the primary G as an error or guessing factor is correct, then its high negative loading on this second-order factor should not be surprising.

It is evident that factor D' has much in common with factor A'. The differentiating feature is the presence of primaries H and F on factor D' and the alternating roles of primaries L and B on the two second-order factors. It thus appears that factor D' might represent the influence of scholastic variables on the activities or abilities defined by factor A'.

CHAPTER V

DISCUSSION OF RESULTS

The aim of the present study was to explore the domain of mathematical ability in general and in particular to study the nature of the numerical and reasoning factors that seem to enter into mathematical thinking. In the hope of measuring something more basic than acquired abilities an effort was made to so select and develop the tests as to minimize, as much as possible, the effect of formal training in mathematics. A scholastic factor, however, did emerge both in the first and in the second order.

A unique development of the study is the splitting of the number factor into two distinct, though correlated, factors one of which, factor A, was identified with what has commonly been called Thurstone's N. The other number factor, factor I, was more difficult to interpret. Its relation to the number factor N is found through the leading saturation of Multiplication on it and through its union with factor A on the second-order number factor B'.

Factor A is best represented by addition and seems to define an ability that is almost on the borderline of reasoning, an automatic sort of manipulation of numbers requiring perhaps only rote memory of various number combinations. Factor I is best represented by multiplication and shows some relation to deductive reasoning by clustering with such tests as Teams and Mixed Series.

Thurstone may have anticipated an outcome of this kind when he stated that "a primary factor may be found to be itself a complex when a part of a domain is investigated with large batteries."[1]

Whether a factor is isolated in the first or in the second order depends upon the selection of the variables in the test battery and on the method of analysis. In the present study only eight of the thirty-six tests were non-mathematical. The mathematical tests varied in nature. That the number factor has become separated in the first order is not so surprising as the finding that Multiplication appears to have so little in common with the other tests of fundamental operations. This curious behavior of multiplication merits further investigation.

The fact that only the two number primaries define the second-order number factor B' and that both primaries have vanishing loadings in the other second-order factors may mean that numerical ability, as defined by the number factor, is not an important element in mathematical ability as here studied. This is not to be understood to mean that computational skills are unimportant. On the contrary, these skills are most necessary to efficient mathematical thought, and the schools must continue to develop them. But it seems that though mathematical ability presumes the number factor the converse is not true.

The nature of several factors strongly suggests that the mathematical processes appear to be mainly processes of eduction, organization, and manipulation of relations. Factors B, D, E, and K, each in its own way, represent abilities that may, in the final analysis, be reduced to a perception of relations and the

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[1]Thurstone, Multiple Factor Analysis, p. 333.

use of this knowledge in the solution of problems.

The above interpretation identifies mathematical thinking with the second principle of noegenesis. If the principles of noegenesis do form the basis of all intellectual activity, then mathematical ability as here studied becomes part of intelligence. This conclusion in itself is neither new nor surprising. The striking thing is that apparently it is the second principle of noegenesis, as distinguished from the third, that is the more closely related to mathematical ability. Factor J which was interpreted as eduction of correlates, the third principle of noegenesis, is practically orthogonal to the rest of the first-order factors and so uncorrelated with them. Its saturations on the second-order factors are all negligible. This finding, of course, may be only the effect of selection in the present study. Final confirmation will have to come from further studies especially designed to investigate this proposition. Such studies may lead to valuable information about the structure of mathematical ability as related to general intelligence.

Factor B was interpreted as flexibility of closure. The element of flexibility that seems to pervade most of the tests on that factor suggests that the more "Gestalt-free" an individual is the better his performance. This interpretation makes it possible to identify factor B with Rimoldi'a A[2] which he defined as a reasoning factor that stressed "the essentially dynamic character of the process, and mainly the plasticity required to perform such an activity" in the complex situations presented in the tests.

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[2]Rimoldi, Central Intellective Factor, p. 83.

Perhaps this flexibility or plasticity of operation is related to the "many-sided nature of thought material" that Duncker discusses in relation to problem solving.[3] One sidedness, or poverty of thought material--an inability to see more than one aspect at a time or to re-structure a concept once formed--is there considered to be the chief distinguishing characteristic of poor thinking and at the same time of a limited kind of mathematical ability.

The second-order analysis was motivated by an expectation of practically useful and fundamentally important results. To some extent this expectation has been realized. It is in the second order that we find some basic clues as to the nature of mathematical ability.

One of the most interesting findings is in connection with second-order factors on which the tests show loadings very similar to their _g_ loadings. The first un-rotated second-order factor I' shows close agreement with the general factor as can be seen from Figure 1. After rotation to simple structure, two second-order factors, A' and D', show similar correspondence.[4] The primary factor D has high loadings on both of these second-order factors. Factor B has a high loading on A' and a moderate loading on D'. The relation of these primaries to the second principle of noegenesis has already been noted.

The factors A' and C' seem to represent the basic characteristics of mathematical thinking, namely, eduction and manipulation of relations, and the ability to abstract from perceptual properties the essential mathematical concepts necessary for the solution of the proposed problems. The fact that it is primary

[3]Duncker, "On Problem-Solving," _Psychological Monographs_, No. 5, 1945. p. 38 ff.

[4]See Figures 2 and 3, pp. 53 and 54.

factors E and K that define the second-order factor C' seems to imply that mathematical ability may to some extent depend upon the degree of fluency and flexibility with which one can work under various forms of restrictions. B' is a number factor. There is no indication that this number factor can be identified with mathematical ability. The clustering of primaries H, F, and L on factor D' suggests that it may be regarded as a second-order scholastic factor which represents the effect of formal training on the innate abilities represented in A'.

Implications for Education

Several of the factors found in this study were interpreted as dealing with relations. The ability to see or to discover relations of similarity or difference, equality or inequality, proved to be an important aspect of the dynamics of mathematical thinking. A considerable portion of the common variance of the tests was explained in terms of factors involving the manipulation of relations. Perhaps more emphasis placed on developing habits of looking for, discovering and analyzing relationships among the various elements of mathematical problems would lead to a more creative approach to the solution of diverse problems of a mathematical nature.

Conclusions

The hypotheses about the nature of mathematical ability proposed by the author have in a great measure been well borne out by the factors isolated in the present study. In particular, the first three points of hypothesis (1)[5] have

[5] See Chapter II, pp. 21-22.

gained factorial support in factors B, D, E, K, and J. Though no explicit factorial evidence was found for the fourth point it is implied in the first three points.

Hypothesis (2) has been partially verified. It has been shown that computational facility as defined by the number factor may not be an important element in mathematical ability as here studied, but the nature of the number factor has not been clearly defined. The splitting of the number factor into two distinct first-order factor suggests new lines of investigation for future research.

The appearance of the verbalization factor lends some support to hypothesis (3).

No direct evidence can be shown for the hypothesis that an aesthetic sense of symmetry, order, and harmony characterizes certain phases of mathematical activity but that is not so surprising. Beauty is such an intangible quality that it can hardly be measured in a purely objective manner.

The tests that did not depend on formal training in particular subject branches proved to be highly instrumental in defining factors that gave the best clues as to the nature of mathematical thinking.

The findings of the present study suggest some problems for future research: Are there two distinct number factors? If so, what is their nature? How do they differ from each other? Does multiplication require abilities not required by the other fundamental operations? Is fluency and flexibility of thought under restrictive conditions a distinctive characteristic of mathematical thinking?

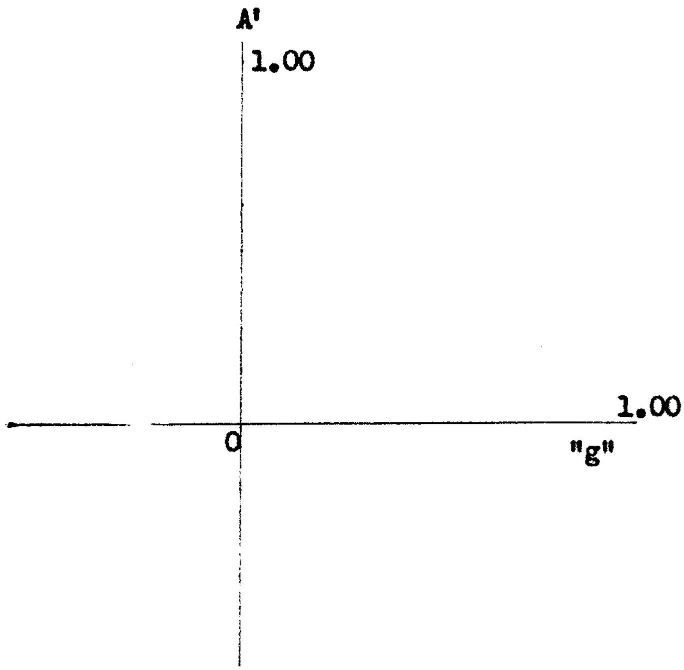

FIGURE 2

PLOT OF "g" LOADINGS AGAINST LOADINGS IN THE FIRST SECOND-ORDER ROTATED FACTOR

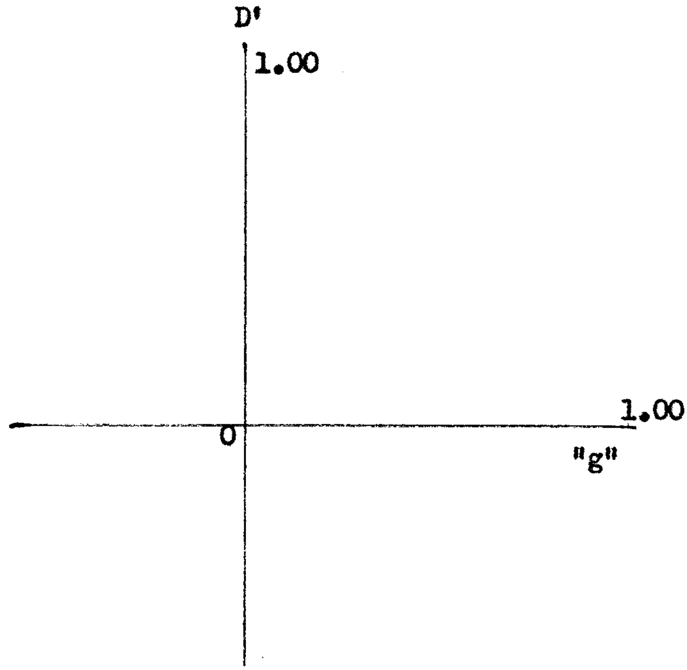

FIGURE 3

PLOT OF "g" LOADINGS AGAINST LOADINGS IN THE FOURTH SECOND-ORDER ROTATED FACTOR

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APPENDIX I

TABLES FOR THE PRELIMINARY STUDY

TABLE I

SUB-BATTERY OF FOURTEEN THURSTONE TESTS FOR THE PRELIMINARY STUDY

1	(T8)[a]	Figure Classification
2	(T26)	Identical Forms
3	(T27)	Pursuit
4	(T29)	Areas
5	(T30)	Number Code
6	(T33)	Multiplication
7	(T35)	Tabular Completion
8	(T37)	Number Series
9	(T38)	Numerical Judgment
10	(T39)	Arithmetical Reasoning
11	(T43)	Code Words
12	(T44)	Pattern Analogies
13	(T45)	Syllogisms
14	(T53)	Hands

Test No. 8 in the PMA Study. Similar code is used for the other tests.

TABLE II

THE CORRELATION MATRIX FOR THE PRELIMINARY STUDY[a]

Test	1	2	3	4	5	6	7	8	9	10	11	12	13	14
1	—													
2	22	—												
3	40	22	—											
4	31	00	43	—										
5	40	22	40	45	—									
6	10	03	29	29	62	—								
7	37	16	22	52	52	47	—							
8	52	16	34	42	40	30	68	—						
9	16	10	28	43	43	52	48	43	—					
10	30	16	25	56	66	48	58	72	64	—				
11	53	40	34	47	66	33	36	58	42	62	—			
12	51	34	46	57	47	10	26	43	46	54	69	—		
13	52	35	43	54	50	38	60	46	33	50	66	43	—	
14	21	03	40	42	50	30	27	17	22	29	37	16	39	—

[a]The decimal point has been omitted for all entries.

TABLE III

THE CENTROID FACTORIAL MATRIX
FOR THE PRELIMINARY STUDY[a]

Test	I	II	III	IV	V	VI	VII	VIII	h^2
1	57	28	25	14	-14	12	-23	05	57
2	31	23	28	20	33	-18	08	12	43
3	56	29	-29	07	-11	-08	-24	21	61
4	68	07	-18	-24	-30	-10	18	08	70
5	78	-11	-20	15	25	27	08	17	85
6	55	-42	-35	18	18	-05	-08	-22	72
7	70	-35	12	18	-29	-17	13	-11	80
8	71	-18	38	-06	-27	13	-20	-08	82
9	62	-32	-09	-25	13	-22	-12	-07	64
10	79	-36	12	-28	05	11	12	06	87
11	81	22	19	-11	27	21	08	-13	89
12	70	34	10	-40	17	-11	-12	22	88
13	76	21	09	27	-10	-11	22	-12	79
14	47	08	-41	16	-14	22	19	08	53

[a]The decimal point has been omitted for all entries.

TABLE IV

THE OBLIQUE FACTOR MATRIX FOR THE PRELIMINARY STUDY[a]

Test	A	B	C	D	E	F	G	H
1	23	01	38	-16	06	-02	-08	03
2	-05	04	05	-05	50	-03	12	-06
3	60	06	08	10	-08	-02	-02	-04
4	46	00	-05	43	-04	07	36	06
5	07	36	17	00	06	53	-06	-05
6	-07	59	-08	03	-06	19	-07	10
7	-01	43	21	06	-03	-06	36	-05
8	00	33	51	02	-12	-05	-05	-06
9	07	50	-02	43	01	-06	00	-03
10	-06	43	24	38	00	25	08	-09
11	-01	03	07	10	42	35	-04	29
12	42	00	05	48	27	-02	00	-02
13	16	02	01	-05	35	09	38	20
14	31	-02	-05	-03	-10	42	14	11

[a]Decimal points have been omitted for all entries.

TABLE V

FINAL TRANSFORMATION MATRIX
FOR THE PRELIMINARY STUDY[a]

	A	B	C	D	E	F	G	H
I	232	305	156	190	143	174	116	070
II	500	-804	-271	-085	437	-027	057	381
III	-475	001	540	-180	317	-286	045	-174
IV	-166	190	211	-828	086	062	070	-108
V	-424	228	-250	043	583	289	-344	021
VI	-240	-183	395	-420	-308	782	-506	195
VII	-236	-364	-372	175	424	432	774	140
VIII	382	133	465	160	-269	033	058	-871

[a]Decimal points have been omitted for all entries.

TABLE VI

REFERENCE VECTOR COSINES FOR
THE PRELIMINARY STUDY

	A	B	C	D	E	F	G	H
A	.996							
B	-.279	1.01						
C	-.114	.374	1.01					
D	.327	.013	-.387	.996				
E	-.289	-.292	-.457	.047	1.01			
F	-.247	-.144	-.016	-.200	.030	.999		
G	.129	-.257	-.333	.294	.330	-.149	1.00	
H	-.114	-.503	-.594	-.094	.358	.235	-.034	1.01

TABLE VII

CORRELATIONS BETWEEN THE PRIMARY FACTORS
FOR THE PRELIMINARY STUDY

	A	B	C	D	E	F	G	H
A	1.00							
B	.62	1.00						
C	.73	.46	1.00					
D	.80	.48	.79	1.00				
E	.69	.43	.61	.61	1.00			
F	.54	.38	.38	.48	.37	1.00		
G	-.10	.17	.10	-.14	-.26	.02	1.00	
H	.64	.64	.77	.65	.36	.26	.21	1.00

APPENDIX II

Description Of The Tests

Ten of the tests are standardized tests. These are listed first. Tests 1 through 5 are the SRA Primary Mental Abilities, Intermediate, Ages 11-17, Form AH, revised 1949 edition published by Science Research Associates, Chicago, Illinois. This Battery of five tests is a shortened and simplified version of the Chicago Tests of Primary Mental Abilities by L.L. Thurstone and T.G. Thurstone, also published by Science Research Associates.

Tests 6 and 7 are the two parts of the California Mathematics Test, Advanced, Grades 9-14, Form AA, 1950 edition by Ernest W. Tiegs and Willis Clark. The tests are published by the California Test Bureau, Los Angeles, California.

Tests 8, 9, and 10 were taken from the Holzinger-Crowder Uni-Factor Tests by Karl J. Holzinger and Norman A. Crowder, Form AM, 1950 edition published by World Book Company, Yonkers-on-Hudson, New York.

Tests 11 through 22 are adaptations of tests used by other investigators in similar studies.

Tests 23 through 36 are new, original tests developed for this study by the writer. The germ ideas for several of these tests were given by Dr. Rimoldi whose generous contribution of time and knowledge is gratefully acknowledged.

Descriptions of the individual tests now follow.

Test No. 1: PMA Number required the addition of four two-digit numbers.

Sample Item:

$$\begin{array}{r} 48 \\ 45 \\ 17 \\ 82 \\ \hline 192 \end{array} \qquad \boxed{R \mid W}$$

The given answer was to be marked as right or wrong. Five practice items and seventy test items were given. Time limit: six minutes.

This test was included in the battery as a reference test for the number factor N.

Test No. 2: PMA Reasoning was a letter series type of test.

Sample Item:

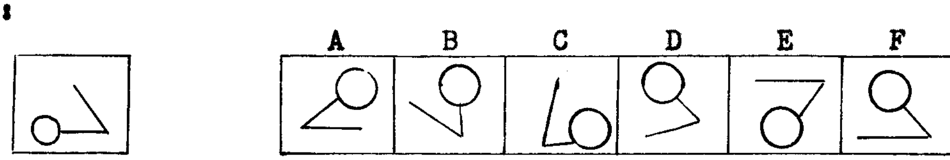
a b x c d x e f x g h x | h | i | j | k | l | x |

The problem was to mark the letter that would come next in the series. Ten practice items and thirty test items were given. Time limit: six minutes.

This test was included as a reference test for reasoning of an inductive nature. In problems of this type it is necessary to discover a rule or principle and then to apply it.

Test No. 3: PMA Space was a simple visualization test involving some kinaesthetic imagery. The test items were rows of figures turned in different directions.

Sample Item:



The task was to mark in each row every figure that was like the first figure even though it had been rotated. Mirror-images - figures made backwardwere not to be marked. The number of figures to mark in a row was not always the same. There were six practice items and twenty test items. Time limit: five minutes.

This test was included as a reference test for the space factor S.

Test No. 4: PMA Verbal-Meaning was a synonym type of test.

Sample Item:

SAFE A. Secure B. Loyal C. Passive D. Young

The task was to mark the word that had the same meaning as the first word. Five practice items and fifty test items were given. Time limit: four minutes.

Test No. 5: PMA Word-Fluency differed from the preceding test in that understanding of the verbal concepts was not involved at all, merely speed and ease in producing words of a certain kind. The task was to write within a given period of time, as many words as possible beginning with the letter s. Time limit: five minutes.

Tests No. 4 and No. 5 were included as reference tests for the verbal factors V and W.

Test No. 6: CMT Reasoning consisted of four sections, namely, Number Concept, Symbols and Rules, Numbers and Equations, and Problems. There were sixty items. Time limit: thirty minutes.

Test No. 7: CMT Fundamentals also consisted of four sections involving the addition, subtraction, multiplication, and division of integers, fractions, and decimals. There were eighty items. Time limit: thirty-eight minutes.

Tests No. 6 and No. 7 were included as external criteria. They are expected to be factorially complex.

<u>Test No. 8</u>: <u>H-C Mixed Series</u> included both number and letter series.

Sample Items:

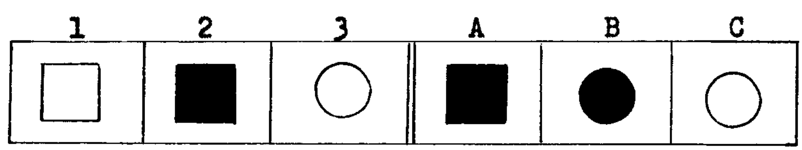
Series	Next Term
A B B C C C D D D	A B F D E
F 1 E 2 D 3 C 4 B	A B 5 1 3
1 3 2 4 3 5 4 6	4 5 6 7 8

The task was to figure out the rule in each series and to use that rule by indicating what the next term should be. Five practice items and forty test items were given. Time limit: seven minutes.

This test was included as a reference test for reasoning factors.

<u>Test No. 9</u>: <u>Figure Changes</u> was a figure-analogies test.

Sample Item:



The problem was to discern the relationship between the first figure and the second figure, and then to select a figure that bore the same relationship to the third figure. Five practice and forty test items were given. Time limit: seven minutes.

Test No. 9 was included as a test of <u>g</u>. It is expected to be factorially complex.

<u>Test No. 10</u>: <u>Teams</u> was a test of syllogistic reasoning. Certain facts were given and the problem was to decide whether each of a series of

conclusions did or did not follow from the given facts.

Sample Item:

Facts	Conclusions
All hurdlers are swimmers	No boxers are swimmers
All swimmers are golfers	All hurdlers are golfers
No swimmers are boxers	Some golfers are swimmers
	No swimmers are hurdlers

The conclusions were to be judged "True" or "False" on the basis of the given facts. The test consisted of four sets of facts and thirty conclusions. Time limit: six minutes.

Test No. 10 was included as a reference test for logical reasoning.

<u>Test No. 11</u>: <u>Number Series</u> was a free-answer test patterned after current number series tests.

Sample Items:

1 3 5 7 9	___
10 9 8 7 6	___
3 5 4 6 5 7	___

The numbers in each row were written according to a rule and the problem seemed to be to discover that rule and to write the number that would follow next in the series. Three practice items and twenty test items were given. Time limit: ten minutes.

<u>Test No. 12</u>: <u>Number Series (2)</u> differed from the preceding test in that each row contained a number that did not belong in the series.

Sample Items:

3	6	9	12	14	15	18	21	___
2	4	6	8	9	10	12	14	___

The problem was to determine which number did not belong in the series and to write it on the blank at the right. Two practice items and twenty test items were given. Time limit: ten minutes.

Test No. 13: Statement to Symbols Translation was patterned after a test used by Kline.[1] The test items were problems of a type commonly found in first-year algebra and involved a direct translation of words into algebraic symbols.

Sample Item: The sum of h and k

a) h x k b) h - k c) h + k d) hk e) h/k

The task was to select from the five given alternatives the one that correctly translated the verbal statement into an algebraic expression. There were twenty test items. Time limit: ten minutes.

This experimental test was included to test the hypothesis that mathematical ability requires a fluency in manipulation of symbols.

Test No. 14: Functional Relationship was patterned after a test used by Weiss.[2] The problems involved cases of direct and inverse variation.

Sample Item:

Y = x - n

If x increases, Y (increases, decreases, remains the same, cannot tell)

If n increases, Y (increases, decreases, remains the same, cannot tell)

The task required the completion of each of the statements at the right by underlining the word or words that made the statement true. There were

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[1]Kline, A Synthesis of Two Factor Analyses.

[2]Weiss, A Factor Analysis of Mathematical Ability.

fifteen items and thirty statements to complete. Time limit: twenty minutes.

This test was designed to test hypotheses 1a and 1c - that mathematical ability involves the seeing of relationships, realizing their implications, drawing conclusions from them, and the ability to handle abstractions.

Test No. 15: Problem Analysis I was patterned after test items included in standard achievement batteries.

Sample Item:

You work after school and Saturdays. You earn \$15 a week. To find out how many weeks it will take to earn \$75, it is necessary to:

1.____ add

2.____ subtract

3.____ multiply

4.____ divide

The task was simply to check the correct answer. Twenty problems were given. Time limit: ten minutes.

Test No. 16: Problem Analysis II was patterned after a test used by Guilford. The test item was a question followed by four statements. Three of the statements contained the information necessary to solve the problem stated in the question. The fourth statement gave irrelevant information.

Sample Item:

Question: How old is Mary?

Statements:
a.___Tom is 4 years older than John.
b.___Mary is 3 years older than Tom.
c.___Tom and John are brothers.
d.___Three years ago, Mary was twice as old as John.

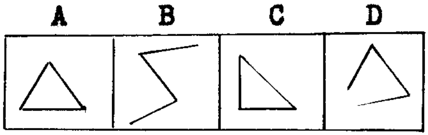
The task was to indicate the irrelevant statement. There were twenty test items. Time limit: ten minutes.

Tests No. 15 and No. 16 were included to test hypothesis 1d - that mathematical ability involves the ability to analyze a situation, distinguish relevant from irrelevant data, and to organize a sequence of steps leading to a solution.

Test No. 17: Figure Grouping was an adaptation of Thurstone's Figure Classification test which was considered a good test of *g*. Each test item consisted of four figures or designs, three of which could be grouped on the basis of a common characteristic. The fourth figure was different.

Sample Item:

A B C D

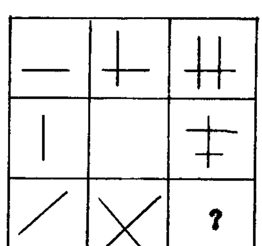


A.___ B.___ C.___ D.___

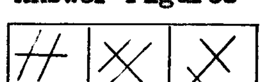
The task was to check the letter that corresponded to the "different" figure.

Test No. 18: Figure Matrix was essentially an adaptation of the Raven Progressive Matrices and similar to a test used by Guilford.

Sample Item:



Answer Figures



A.___ B.___ C.___

In each matrix one or more cells were left empty and the problem was to indicate the answer figure that would correctly fill the square indicated by the question mark. There were twenty-four items. Time limit: twenty minutes.

Tests No. 17 and No. 18 were included to test hypotheses 1a and 1b and to some extent hypothesis 4. They were expected to be factorially complex.

<u>Test No. 19</u>: <u>Addition</u> consisted of problems requiring the addition of two, three, or four single-digit or two-digit numbers.

Sample Items:

Add:

				8	98
		1	37	0	89
7	45	5	95	3	72
5	39	3	24	7	33

There were seventy items. Time limit: six minutes.

<u>Test No. 20</u>: <u>Subtraction</u> required the subtraction of one-digit, two digit, or three-digit numbers.

Sample Items:

Subtract:

15	44	462
8	19	174

There were eighty items. Time limit: six minutes.

<u>Test No. 21</u>: <u>Multiplication</u> required the multiplication of one, two, three, or four-digit numbers by a one-digit number.

Sample Items:

Multiply:

8	47	674	9171
6	3	5	9

There were seventy items. Time limit: twelve minutes.

<u>Test No. 22</u>: <u>Division</u> consisted of two parts. The first part required the division of two-digit and three-digit numbers by a one-digit number. The

divisions were exact. In the second part the divisions were not exact but only the remainder was asked for.

Sample Items:

Part 1: Perform the division

$7\overline{)56}$ $4\overline{)52}$ $3\overline{)693}$ $6\overline{)498}$

Part 2: Perform the division and write only the remainder.

$7\overline{)17}$ R____ $3\overline{)16}$ R____ $9\overline{)75}$ R____

There were sixty items. Time limit: six minutes.

Test No. 23: Conditions 1 was a test of logical reasoning in very abstract form. The test consisted of a set of equations defining relations between certain quantities. For each set of conditions some conclusions were given and the problem was to judge whether or not the conclusions were true under the given conditions.

Sample Items:

Conditions	Conclusions
Given that $a=b$, $b=f$, $f \neq d$	Then: 1. $a = d$ ____ 2. $f = a$ ____
Given that $a = b = c$, $b > m$	Then: 1. $a = m$ ____ 2. $a = c$ ____ 3. $c = m$ ____ 4. $m < a$ ____

The task was to place a check mark on the blank if the conclusion was true under the given conditions. There were six sets of conditions with thirty conclusions. Time limit: fifteen minutes.

Test No. 24: Conditions 2 was similar to test No. 23 except that the student had to indicate the relation which would express a true conclusion under the given conditions.

Sample Item:

Conditions	Conclusions
Given that $a = b$ $b > c$	Then: 1. c____b 2. c____a 3. a____c

The required task was to write a symbol on the blank between the letters such that the conclusion would be true. The usual symbols, =, >, and < were permitted. If the relation between the two quantities could not be determined on the basis of the given facts the symbol 0 was to be written. There were eight sets of conditions and thirty conclusions. Time limit: twenty minutes.

Tests No. 23 and No. 24 were included to test hypotheses 1a, 1b, and 1c. It will be interesting to find out whether these two tests require the same kind of ability as the verbal test in syllogistic reasoning, Test No. 10.

Test No. 25: Fluency With Mathematical Expressions

Sample Items:

In how many different ways can you write that 3 times the sum of c and d is to be divided by m?

The purpose of this test was to determine, in some measure, to what extent fluency with verbal material, as presented in Test No. 5, compared with fluency and ease in handling more or less abstract mathematical concepts. It must be remembered that Test No. 5, Word-Fluency, required no verbal comprehension whatever, whereas this test calls for a certain amount of understanding of both verbal and mathematical concepts. There were eight items.

Time limit: two minutes per item.

Test No. 26: Quantitative Relationship

Sample Items:

6	8
5(2)	8 + 10
3 + 7 - 5	3 - 7 + 5
3 x 7 x 5	5 x 3 x 7

The required task was to determine the relationship between the two given expressions and to place the proper mathematical sign (>, =, or <) between them. There were twenty-five items. Time limit: ten minutes.

Test No. 27: Numerical Inequalities required number-operations with inequalities. In each row the first problem was solved and the student was instructed to solve the remaining problems in the row in the same way as the first one, that is, either by addition, or subtraction, or multiplication, or division.

Sample Items:

1. $\begin{array}{c} 7 > 4 \\ \underline{2 = 2} \\ 9 > 6 \end{array} \qquad \begin{array}{c} 10 < 15 \\ \underline{8 = 8} \end{array} \qquad \begin{array}{c} 5 = 5 \\ \underline{6 > 5} \end{array} \qquad \begin{array}{c} 7 = 7 \\ \underline{5 < 6} \end{array}$

2. $\begin{array}{c} 8 < 12 \\ \underline{4 = 4} \\ 2 < 3 \end{array} \qquad \begin{array}{c} 18 = 18 \\ \underline{6 > 3} \end{array} \qquad \begin{array}{c} 20 > 20 \\ \underline{4 < 5} \end{array} \qquad \begin{array}{c} 16 < 24 \\ \underline{8 = 8} \end{array}$

There were five rows containing twenty test items in all. Time limit: ten minutes.

Test No. 28: Algebraic Inequalities was similar to the preceding test except that the numbers were replaced by letters.

Sample Item: $\dfrac{\begin{array}{c} a > b \\ c = c \end{array}}{(a+c) > (b+c)}$ $\qquad \dfrac{b = d}{c > d} \qquad \dfrac{x > y}{f < e} \qquad \dfrac{x = x}{r < t} \qquad \dfrac{a < b}{c < c}$

As in the preceding test, the student was instructed to solve all the problems in a row in the same way that the first one was solved. If the relation between the quantities could not be determined, the symbol 0 was to be used. There were twenty test items. Time limit: ten minutes.

Tests No. 27 and No. 28 and to some extent Test No. 26 were designed to test hypothesis 1c and to investigate the role of the number factor in mathematical tasks.

<u>Test No. 29</u>: <u>General Expressions</u> required the student to discover the relation between numbers in a sequence and to write a general expression for that relationship.

Sample Items:

$3 + 4 = 4 + 3; \qquad 9 + 7 = 7 + 9; \qquad 2 + 8 = 8 + 2$

The general expression is: $a + b = b + a$

$3/6 \qquad 7/14 \qquad 4/8 \qquad 6/12 \qquad 10/20$

The general expression is: $n/2n$

$\dfrac{3 + 4}{5} \qquad \dfrac{5 + 6}{5} \qquad \dfrac{7 + 8}{5} \qquad \dfrac{8 + 9}{5}$

The general expression is: $\dfrac{n + (n + 1)}{5}$

There were twenty items. Time limit: twenty minutes.

<u>Test No. 30</u>: <u>Number Oddities</u> was a rather unusual test designed to test hypothesis 4 and its relation to hypothesis 1a.

Sample Item:

Observe that:

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$

Now write the next two lines: ______________________________

____

There were twenty such items. Time limit: thirty minutes.

This test was inspired by Hardy's[3] description of a mathematician as a "maker of patterns," patterns more beautiful and lasting than those of an artist or a poet because patterns made of ideas, rather than mere colors or words. Poincare[4] too speaks of an aesthetic sense of harmony and order.

Test No. 31: Number Relations was a test of fluency and originality in handling number relations.

Sample Items:

Instructions: Given a set of numbers, how many different relations can you set up using just the given numbers and any mathematical signs you need?

1. Given: 2, 4, 6, 24

   Possible Answers: $2 + 4 = 6$

   $6 - 2 = 4$

   $24/6 = 4$

2. Given: 2, 2, 3, 4, 8

   Possible Answers: $2^2 = 4$

   $4 \times 2 = 2^3$

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[3] G. Hardy, A Mathematicians Apology, Cambridge, Eng., 1940.

[4] Henri Poincare, Science and Method, New York, 1952.

There were ten test items. Time limit: two minutes per item.

Test No. 32: Number Fluency required the student to write quickly as many numbers as he could think of that would satisfy certain given conditions.

Sample Items:

1. Write odd numbers that are divisible by 7.
2. Write multiples of 3 such that no two are in the same decade.

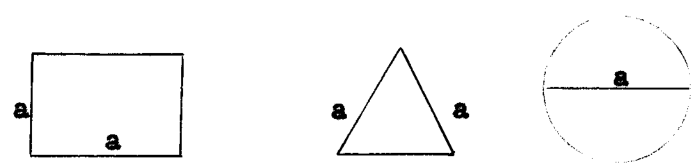
There were ten items. In each case several examples of acceptable answers were given.

Tests No. 31 and No. 32 together with Test No. 25 were partly designed as the mathematical counterparts of Test No. 5, Word-Fluency.

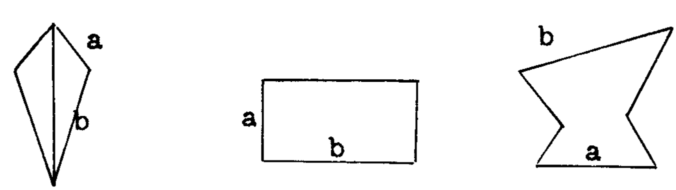
Test No. 33: Formulas and Figures required that the student match an algebraic expression, usually part of a formula, with a geometric figure.

Sample Items:

Instructions: Match the algebraic expression with the figure that best describes it.



1. a^2 A. _____ B. _____ C. _____



2. ab A. _____ B. _____ C. _____

There were twenty items. Time limit: fifteen minutes. The test was designed to test hypothesis 1a.

Test No. 34: Mixed Operations was a simple number-operations test included to study the effect of flexibility on the number factor. The solutions required a quick shifting from one operation to another.

Sample Items:

8 + 4, -5, x2, +3, x6, +5, -6, +1 =	0	1	2	3
7 x 3, -1, +4, x3, +7, -4, +6, +3 =	5	6	7	8

The student was instructed to circle the correct answer from the numbers given at the right. There were twenty items. Time limit: six minutes.

Test No. 35: Missing Number was a simple number-operations test. The test item was an incomplete equation which the student had to complete by supplying the missing number.

Sample Items:

1. 3 + ? = 10   1. _____
2. ? + 7 = 16   2. _____
3. 18 - (5 - ?) = 14   3. _____

There were twenty items. Time limit: ten minutes.

Test No. 36: Missing Sign was a companion test to Missing Number. In this test the student was required to supply the mathematical sign that would make an incomplete equation true.

Sample Items:

1. 3 ? 7 - (2 x 4) = 2   1. _____
2. 15 ? 4 = 60   2. _____
3. 8 - 2 ? 3 = 9   3. _____

Twenty items were given. Time limit: ten minutes.

Tests No. 35 and No. 36 were designed to discover whether there is a difference in the type of reasoning that is done when performing the two apparently similar tasks - supplying a number or supplying a sign of operation.

APPENDIX III

TABLES FOR THE MAIN STUDY

TABLE VIII

I.Q. FREQUENCY DISTRIBUTION
OF EXPERIMENTAL GROUP
FOR THE MAIN STUDY

I.Q.	Frequency
134 - 136	1
131 - 133	2
128 - 130	3
125 - 127	5
122 - 124	6
119 - 121	7
116 - 118	17
113 - 115	20
110 - 112	17
107 - 109	22
104 - 106	20
101 - 103	15
98 - 100	6
95 - 97	4
92 - 94	2
89 - 91	2
86 - 88	1

TABLE IX

AGE DISTRIBUTION OF SUBJECTS
FOR THE MAIN STUDY

Age in Months	Frequency
210 - 211	1
208 - 209	0
206 - 207	0
204 - 205	0
202 - 203	2
200 - 201	2
198 - 199	17
196 - 197	15
194 - 195	16
192 - 193	25
190 - 191	35
188 - 189	25
186 - 187	9
184 - 185	3

TABLE X

THE TEST BATTERY FOR THE MAIN STUDY

Number	Name of Test	Number of Items	Time Limit (minutes)	Scoring Formula
1	PMA Number	70	6	R-W
2	PMA Reasoning	30	6	R
3	PMA Space	20	5	R-W
4	PMA Verbal-Meaning	50	4	R
5	PMA Word-Fluency	--	5	R
6	CMT Reasoning	60	30	R
7	CMT Fundamentals	80	38	R
8	Mixed Series	40	7	R
9	Figure Changes	40	7	R
10	Teams	30	6	R-W
11	Number Series	20	10	R
12	Number Series (2)	20	15	R
13	Statement Translation	20	10	R
14	Functional Relationship	30	20	R
15	Problem Analysis I	20	10	R
16	Problem Analysis II	20	10	R
17	Figure Grouping	30	15	R
18	Figure Matrix	24	20	R
19	Addition	70	6	R
20	Subtraction	80	6	R
21	Multiplication	70	12	R
22	Division	60	6	R
23	Conditions 1	30	15	R
24	Conditions 2	30	20	R
25	Fluency With Mathematical Expressions	8	16	R
26	Quantitative Relationship	25	10	R
27	Numerical Inequalities	20	10	R
28	Algebraic Inequalities	20	10	R
29	General Expressions	20	20	R
30	Number Oddities	20	30	R
31	Number Relations	10	20	R
32	Number Fluency	10	20	R
33	Formulas and Figures	20	15	R
34	Mixed Operations	20	6	R
35	Missing Number	20	10	R
36	Missing Sign	20	10	R

TABLE XI

MEANS AND STANDARD DEVIATIONS OF RAW SCORES,
STANDARD ERRORS AND RELIABILITIES
FOR THE MAIN STUDY

Test	Means	S.E. of M	D.D.	S.E. of S.D.	Reliability
1	23.20	.530	6.45	.370	.92
2	20.53	.408	5.00	.288	.88
3	18.87	1.110	13.59	.785	.92
4	31.14	.698	8.56	.493	.91
5	44.46	.748	9.17	.528	.77
6	40.45	.582	7.14	.528	.75
7	64.10	.751	9.21	.531	.82
8	24.34	.403	4.95	.285	.73
9	24.07	.559	6.85	.395	.87
10	17.58	.591	7.24	.417	.80
11	16.98	.249	3.06	.176	.81
12	13.07	.255	3.13	.180	.71
13	14.98	.235	2.88	.166	.77
14	18.22	.379	4.65	.268	.80
15	16.06	.211	2.59	.149	.68
16	14.88	.196	2.40	.138	.60
17	23.01	.180	2.20	.137	.30
18	20.65	.196	2.40	.138	.54
19	68.47	.144	1.77	.102	.50
20	77.27	.414	5.07	.293	.88
21	66.94	.348	4.26	.246	.72
22	58.40	.278	3.41	.196	.80
23	24.66	.334	4.10	.236	.81
24	22.24	.403	4.94	.285	.84
25	14.94	.746	9.14	.527	.90
26	18.25	.334	4.10	.236	.80
27	18.98	.440	5.39	.311	.78
28	10.13	.228	2.80	.161	.68
29	7.00	.415	5.09	.293	.92
30	28.50	.498	6.10	.352	.80
31	37.13	.995	12.19	.703	.66
32	88.38	2.460	30.12	1.740	.85
33	10.23	.214	2.63	.151	.44
34	18.45	.210	2.57	.148	.90
35	15.13	.239	2.93	.169	.71
36	14.81	.245	3.00	.173	.82

TABLE XII

PRODUCT-MOMENT CORRELATIONS BETWEEN THE TESTS FOR THE MAIN STUDY[a]

Test	1	2	3	4	5	6	7	8	9	10	11	12
1												
2	46											
3	28	41										
4	52	30	29									
5	41	28	20	26								
6	50	37	34	41	31							
7	56	42	41	41	27	74						
8	42	44	30	23	16	30	38					
9	30	53	33	20	19	22	29	42				
10	34	30	26	29	18	46	48	42	25			
11	37	35	26	20	16	39	38	27	25	33		
12	39	44	34	28	19	56	50	28	50	42	49	
13	38	39	34	36	32	68	60	24	23	42	35	47
14	41	39	23	25	29	52	57	33	29	30	31	43
15	41	18	21	30	26	73	59	25	21	39	36	46
16	29	23	28	23	23	43	43	14	11	38	35	48
17	23	27	16	16	15	34	36	27	19	26	29	29
18	21	46	35	20	12	40	36	45	24	26	35	48
19	65	26	10	37	16	41	41	31	13	19	24	34
20	50	19	17	20	12	41	45	31	01	11	24	29
21	56	11	14	26	27	36	45	20	06	30	21	22
22	60	27	15	19	15	42	52	21	13	31	31	33
23	24	35	25	20	24	53	50	33	29	48	45	45
24	31	42	36	25	29	49	48	32	32	52	47	53
25	-02	14	07	-02	01	25	16	02	09	07	07	18
26	46	43	36	42	35	66	62	36	30	50	45	61
27	33	18	02	11	05	23	30	24	17	30	28	30
28	27	22	13	21	24	35	49	09	28	31	32	33
29	26	50	40	34	27	50	48	28	36	44	44	48
30	45	69	49	29	22	51	55	33	41	36	37	49
31	46	39	33	32	27	60	58	34	27	39	36	55
32	57	47	27	38	36	51	43	41	20	36	40	42
33	21	26	21	13	26	27	41	28	14	20	33	31
34	50	25	22	29	13	34	40	31	04	25	37	40
35	52	54	45	39	29	56	58	37	37	42	47	58
36	49	49	37	38	21	44	57	39	37	26	39	46

[a]Decimal points have been omitted for all entries.

TABLE XII (continued)

PRODUCT-MOMENT CORRELATIONS BETWEEN THE TESTS FOR THE MAIN STUDY[a]

Test	13	14	15	16	17	18	19	20	21	22	23	24
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												
13												
14	49											
15	47	54										
16	47	29	41									
17	36	16	26	32								
18	35	21	28	28	27							
19	26	31	26	12	18	21						
20	27	24	33	21	06	24	35					
21	18	19	30	17	15	17	37	24				
22	33	30	32	24	16	21	47	33	48			
23	58	37	43	49	37	32	19	18	16	22		
24	55	44	44	44	22	39	24	13	05	23	61	
25	35	10	19	11	05	-01	-08	-12	-05	01	21	10
26	65	56	60	45	36	39	35	40	38	40	55	60
27	16	16	15	11	16	21	19	21	16	09	21	21
28	46	61	32	33	20	22	15	16	17	32	31	27
29	57	41	47	40	34	41	23	18	14	26	51	55
30	49	38	41	41	34	47	21	21	32	27	38	44
31	58	49	51	45	20	42	28	39	36	29	47	58
32	36	37	39	26	25	37	23	34	19	29	44	43
33	30	27	29	32	13	35	18	19	23	14	35	35
34	39	23	36	26	28	24	37	35	29	40	30	25
35	48	39	43	31	48	47	33	25	18	27	53	44
36	38	41	47	33	21	46	27	32	30	26	36	39

[a]Decimal points have been omitted for all entries.

TABLE XII (continued)

PRODUCT-MOMENT CORRELATIONS BETWEEN THE TESTS FOR THE MAIN STUDY[a]

Test	25	26	27	28	29	30	31	32	33	34	35	36
1												
2												
3												
4												
5												
6												
7												
8												
9												
10												
11												
12												
13												
14												
15												
16												
17												
18												
19												
20												
21												
22												
23												
24												
25												
26	23											
27	-19	32										
28	06	36	12									
29	25	59	14	36								
30	17	50	24	28	40							
31	10	59	24	31	44	45						
32	02	44	31	19	45	36	47					
33	06	35	10	37	31	27	31	16				
34	08	42	19	24	39	24	33	31	18			
35	09	56	27	24	54	57	52	49	19	42		
36	10	47	29	25	40	48	40	34	36	39	58	

[a]Decimal points have been omitted for all entries.

TABLE XIII

FREQUENCY DISTRIBUTION OF THE TEST INTER-CORRELATIONS FOR THE MAIN STUDY

Correlation	Frequency
.73 - .77	2
.68 - .72	2
.63 - .67	3
.58 - .62	19
.53 - .57	23
.48 - .52	43
.43 - .47	56
.38 - .42	73
.33 - .37	86
.28 - .32	84
.23 - .27	86
.18 - .22	66
.13 - .17	42
.08 - .12	21
.03 - .07	11
-.01 - .02	7
-.06 - (-.02)	3
-.11 - (-.07)	1
-.16 - (-.12)	1
-.21 - (-.19)	1
	630

TABLE XIV

THE CENTROID FACTOR MATRIX
FOR THE MAIN STUDY[a]

Test	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII	h^2_j
1	72	-58	21	20	06	-10	04	-16	-22	-13	12	05	1.00[b]
2	63	21	38	28	-07	-06	17	06	-11	07	-06	15	76
3	48	18	16	13	12	-14	14	19	03	06	08	-15	43
4	48	-18	08	06	15	07	17	08	-07	-19	10	-24	45
5	39	-03	-09	12	-07	-08	20	08	-11	-24	01	-03	31
6	79	-12	-24	-13	16	08	12	10	11	18	02	08	81
7	82	-16	-17	12	10	05	-11	09	20	04	06	14	83
8	54	-08	32	06	-28	-08	08	-17	26	09	-30	-06	68
9	45	26	29	33	-20	22	13	-19	10	06	11	-05	63
10	58	10	-14	-10	-07	-11	08	-29	18	-11	16	-02	56
11	58	12	08	-14	-05	-04	-20	-08	-07	-10	-04	05	45
12	73	19	11	-15	-05	16	-17	-08	-10	19	21	-14	77
13	72	16	-29	-03	13	06	09	03	-13	03	-03	08	69
14	62	-08	-26	19	-24	29	08	12	-07	05	03	07	67
15	66	-13	-30	-14	00	14	07	08	16	02	-03	-10	63
16	54	19	-20	-12	06	-18	-11	06	-05	-05	07	-03	45
17	43	17	09	-14	22	05	-07	-06	12	-17	-02	14	36
18	55	16	24	-08	-07	-11	-14	21	11	13	-09	-09	51
19	47	-40	09	05	06	04	-09	-19	-13	08	04	-02	46
20	43	-41	05	-15	-05	-04	-11	18	-07	20	-10	00	48
21	42	-42	-09	18	15	-24	-20	-15	18	-05	10	-06	58
22	50	-29	-08	10	13	-11	-22	-21	-17	14	04	09	54
23	65	28	-17	-21	-04	-06	02	-10	06	-10	-08	09	62
24	67	27	-14	-15	-21	-13	15	-06	-06	04	07	-13	68
25	16	29	-25	08	27	12	22	-10	-04	15	-30	06	44
26	80	03	-15	-12	-01	10	05	-04	-05	06	-01	-06	70
27	33	-11	20	-24	-22	08	-12	-08	14	-09	10	11	35
28	49	08	-31	24	-18	22	-28	13	-13	-18	11	15	66
29	68	22	-05	-07	05	08	07	02	-12	-06	-14	-12	62
30	68	21	19	20	15	-15	04	13	12	14	13	18	75
31	71	-03	-13	-14	-10	-09	07	11	-01	15	12	-06	61
32	62	-16	14	-22	-20	-13	28	07	-11	-16	-04	18	65
33	45	11	-14	17	-21	-15	-30	13	04	-06	-21	-17	52
34	53	-18	09	-15	22	04	-19	-10	-21	-05	-21	-06	53
35	74	09	28	-13	16	12	09	05	08	-11	04	06	72
36	66	-04	20	09	04	08	-08	15	14	-02	-08	-10	55

[a]Decimal points have been omitted (except for last entry in row 1).
[b]Within limits of error

TABLE XV

THE OBLIQUE FACTOR MATRIX
FOR THE MAIN STUDY[a]

Test	A	B	C	D	E	F	G	H	I	J	K	L
1	36	01	05	-02	08	06	-01	14	16	20	00	24
2	10	44	-03	-05	-06	-08	06	04	-09	30	05	20
3	-02	39	-15	02	00	-08	-01	22	-03	-02	05	02
4	02	02	01	-02	00	01	-01	41	01	07	00	08
5	-11	-04	-08	-01	01	01	18	12	02	05	-02	32
6	11	13	04	20	03	42	-14	-02	12	-06	-07	-03
7	04	18	15	00	-01	40	12	-01	22	03	03	-03
8	-06	08	08	03	02	00	-07	-01	33	33	43	06
9	-02	31	-02	04	00	00	00	10	16	57	02	-10
10	-10	-01	00	35	25	-03	00	00	40	02	-10	01
11	17	01	17	18	20	-07	20	12	-01	-02	13	-01
12	35	23	-02	33	32	03	-02	21	-06	14	01	-36
13	10	05	-02	32	-03	23	07	03	-03	-02	-09	05
14	05	01	-05	02	00	48	18	-04	-01	34	-01	06
15	-03	-08	05	16	05	42	-01	10	17	00	06	-06
16	06	07	-05	26	17	-02	19	09	00	-24	01	00
17	-05	08	34	07	02	-01	06	15	07	-09	-03	-01
18	09	32	04	-01	13	00	09	15	-07	-05	32	-10
19	38	-03	05	11	05	04	-10	07	12	16	05	-07
20	34	02	03	-05	13	22	-06	-01	-08	-06	22	00
21	09	01	02	03	00	02	06	01	43	-08	11	-02
22	43	01	-02	23	02	02	01	-08	14	01	02	-07
23	-06	-04	10	34	15	05	13	01	14	-08	03	08
24	02	08	-24	42	30	-03	03	04	08	02	01	04
25	-04	-01	-02	30	-43	15	-12	-05	00	05	-02	-01
26	16	02	01	31	11	21	00	12	06	08	05	-05
27	05	-02	26	-04	36	05	02	01	10	08	03	-02
28	03	-03	09	-04	03	33	53	00	-09	16	-01	03
29	05	03	03	28	-02	03	12	27	-08	05	13	01
30	04	58	00	00	00	03	01	-02	06	02	-05	02
31	14	16	-17	23	27	17	-03	01	05	-05	00	-01
32	04	-02	07	02	28	02	00	-01	-02	01	-01	43
33	-02	-01	-06	03	-02	05	46	10	06	00	46	-01
34	37	-14	23	17	-05	-05	04	27	-07	-05	23	-05
35	04	23	29	01	10	04	-04	29	-01	06	-01	02
36	03	21	17	-11	-04	14	11	28	04	13	26	-06

[a]Decimal points have been omitted for all entries.

TABLE XVI

FINAL TRANSFORMATION MATRIX
FOR THE MAIN STUDY[a]

	A	B	C	D	E	F	G	H	I	J	K	L
I	15	16	08	20	11	15	10	15	11	10	11	03
II	-24	30	-08	29	-02	-25	20	06	-20	-04	-07	-15
III	18	41	27	-36	11	-44	-19	30	-19	24	18	-03
IV	-14	31	-24	-30	-57	05	31	-06	18	50	09	11
V	11	17	25	07	-48	-09	-28	32	-04	-41	-23	-18
VI	03	-18	44	-17	-20	57	-06	27	-23	58	-13	-37
VII	-35	04	-31	08	-12	05	-53	-07	10	20	-38	49
VIII	-18	34	-04	-64	-03	38	28	16	-59	-33	12	19
IX	-66	18	30	-26	-04	32	-13	-12	69	-02	14	-19
X	50	48	-44	30	-03	24	-58	-45	-06	10	02	-52
XI	06	40	-26	01	61	-09	-12	-00	05	-06	-73	-22
XII	02	14	39	-22	-01	25	07	-69	-08	-05	-40	40

[a]Decimal points have been omitted for all entries.

TABLE XVII

REFERENCE VECTOR COSINES FOR THE MAIN STUDY[a]

	A	B	C	D	E	F	G	H	I	J	K	L
A												
B	07											
C	-16	-25										
D	35	-22	-39									
E	16	05	-15	17								
F	-17	-03	20	-30	-22							
G	-22	-22	20	-33	-05	-02						
H	-05	-14	21	-12	-10	-26	16					
I	-42	-12	00	22	-03	-01	-19	-24				
J	02	-01	00	-09	-22	21	-06	00	13			
K	00	-21	10	-14	-29	-06	34	25	03	07		
L	-37	-21	00	-29	-08	-11	25	-20	-05	-10	-09	

[a]Decimal points have been omitted for all entries.

TABLE XVIII

SUMMARY OF OBLIQUE FACTOR MATRIX
FOR THE MAIN STUDY[a]

TESTS		FACTORS											
		A	B	C	D	E	F	G	H	I	J	K	L
22	Division	43			23								
19	Addition	38											
34	Mixed Operations	37		23					27			23	
1	PMA Number	36											24
12	Number Series 2	35	23		33	32			21				-36
20	Subtraction	34					22						
30	Number Oddities		58										
2	PMA Reasoning		44								30		
3	PMA SPACE		39						22				
18	Figure Matrix		32									32	
9	Figure Changes		31								57		
35	Missing Number		23	29					29				
36	Missing Sign		21						28			26	
17	Figure Grouping			34									
27	Numerical Inequalities			26		36							
24	Conditions 2			-24	42	30							
10	Teams				35	25				40			
23	Conditions 1				34								
13	Statement Translation				32		23						
26	Quantitative Relations				31		21						
25	Fluency With Math. Exp.				31	-43							
29	General Expressions				28				27				
16	Problem Analysis II				26								
31	Number Relations				23	27							
32	Number Fluency					28							43
11	Number Series 1					20							
14	Functional Relationship						48				34		
6	CMT Reasoning						42						
15	Problem Analysis I						42						
7	CMT Fundamentals						40			22			
28	Algebraic Inequalities						33	53					
33	Formulas and Figures							46				46	
4	PMA Verbal-Meaning								41				
21	Multiplication									43			
8	Mixed Series									33	33	43	
5	PMA Word-Fluency												32

[a]Decimal points have been omitted for all entries and loadings less than 20 are not listed.

TABLE XIX

CORRELATIONS BETWEEN THE PRIMARY FACTORS
FOR THE MAIN STUDY

	A	B	C	D	E	F	G	H	I	J	K	L
A	1.00											
B	.10	1.00										
C	-.07	.40	1.00									
D	-.21	.57	.48	1.00								
E	-.09	.14	.13	.13	1.00							
F	.29	.36	.01	.37	.23	1.00						
G	.25	.20	-.04	.15	-.11	.12	1.00					
H	.37	.32	-.06	.20	.12	.53	.09	1.00				
I	.66	.14	-.13	-.20	-.04	.26	.28	.43	1.00			
J	-.18	.06	.14	.16	.19	-.14	.00	-.09	-.23	1.00		
K	-.17	.25	.17	.27	.34	.15	-.32	-.07	-.20	.05	1.00	
L	.37	.51	.23	.43	.20	.49	-.02	.49	.27	.04	.23	1.00

TABLE XX

LOADINGS OF THE PRIMARIES IN THE CENTROIDS OF THE SECOND-ORDER FOR THE MAIN STUDY[a]

	I'	II'	III'	IV'
A	07	76	11	03
B	74	15	06	-03
C	58	-18	-12	-36
D	82	-21	-17	17
E	22	-14	42	13
F	44	30	32	39
G	10	39	-30	-02
H	34	42	09	52
I	02	82	20	00
J	15	-22	04	-22
K	33	-38	50	-13
L	65	25	23	22

[a]Decimal points have been omitted for all entries.

TABLE XXI

ROTATED FACTORIAL MATRIX FOR THE PRIMARIES IN THE SECOND-ORDER FOR THE MAIN STUDY[a]

	A'	B'	C'	D'
A	05	55	-04	04
B	56	12	12	36
C	65	03	09	03
D	60	-34	-09	60
E	-09	-02	40	15
F	04	08	18	51
G	25	16	-34	09
H	00	00	-11	60
I	00	65	04	-02
J	18	-01	16	-11
K	07	-01	61	00
L	31	10	17	50

[a]Decimal points have been omitted for all entries.

TABLE XXII

TRANSFORMATION MATRIX FOR THE SECOND-ORDER FOR THE MAIN STUDY[a]

	A'	B'	C'	D'
I'	76	-03	11	53
II'	07	69	-18	00
III'	-43	38	93	-16
IV'	-48	-61	-29	83

[a]Decimal points have been omitted for all entries.

TABLE XXIII

CORRELATIONS BETWEEN THE SECOND-ORDER FACTORS FOR THE MAIN STUDY

	A'	B'	C'	D'
A'	1.00			
B'	. .16	1.00		
C'	-.19	.41	1.00	
D'	.08	-.58	-.34	1.00

TABLE XXIV

LOADINGS OF THE TESTS IN THE ORTHOGONAL SECOND-ORDER FACTORS FOR THE MAIN STUDY[a]

Test	I'	II'	III'	IV'
1	26	64	24	13
2	68	10	10	-01
3	56	09	07	20
4	35	29	11	31
5	35	17	06	24
6	70	32	19	24
7	64	49	20	14
8	34	09	41	-15
9	30	10	07	-04
10	26	29	05	11
11	46	14	03	00
12	52	18	06	13
13	67	21	-05	27
14	50	23	18	22
15	53	23	23	27
16	51	15	-04	20
17	41	09	-09	-03
18	65	-03	23	01
19	09	49	16	04
20	37	22	35	07
21	01	65	18	04
22	12	56	04	03
23	58	06	00	10
24	56	04	09	26
25	23	-09	-20	08
26	62	24	13	23
27	19	08	24	-11
28	39	26	-06	14
29	67	02	-02	22
30	72	24	06	03
31	61	22	22	26
32	61	11	27	13
33	40	09	14	07
34	33	27	04	06
35	72	15	10	08
36	61	20	23	07

[a]Decimal points have been omitted for all entries.

TABLE XXV

LOADINGS OF THE TESTS IN THE SECOND-ORDER ROTATED FACTORS FOR THE MAIN STUDY[a]

Test	A'	B'	C'	D'
1	27	44	09	21
2	52	09	15	34
3	54	-04	05	45
4	44	04	00	43
5	40	-02	-01	38
6	68	12	12	55
7	56	31	12	41
8	22	30	45	-01
9	18	11	08	09
10	26	15	00	22
11	35	09	05	24
12	47	05	04	38
13	64	-06	-09	59
14	51	08	11	42
15	56	07	14	47
16	49	-05	-07	45
17	28	04	-05	20
18	53	04	28	32
19	09	37	06	06
20	36	23	31	20
21	03	49	04	01
22	09	38	-06	08
23	50	-04	02	40
24	57	-11	06	50
25	20	-20	17	22
26	61	02	08	51
27	11	21	27	03
28	36	06	-10	33
29	62	-15	-02	54
30	57	15	08	40
31	63	06	16	51
32	56	08	26	39
33	35	06	14	25
34	28	15	00	22
35	60	07	12	44
36	52	16	22	35

[a]Decimal points have been omitted for all entries.

TABLE XXVI

"G" LOADINGS OF THE TESTS FOR THE MAIN STUDY

Test	"g"-loading
1	.49
2	.39
3	.22
4	.23
5	.15
6	.62
7	.67
8	.27
9	.19
10	.33
11	.34
12	.52
13	.52
14	.38
15	.42
16	.29
17	.18
18	.29
19	.22
20	.19
21	.16
22	.24
23	.42
24	.43
25	.03
26	.65
27	.12
28	.22
29	.46
30	.46
31	.50
32	.38
33	.19
34	.27
35	.54
36	.43

TABLE XXVII

SECOND-ORDER FACTOR PATTERN
FOR THE MAIN STUDY[a]

Primaries	Second-Order Factors			
	A'	B'	C'	D'
C	65			
D	60	-34		60
B	56			34
L	31			50
I		65		
A		55		
K			61	
E			55	
G			-34	
H				60
F				51

[a]Decimal points have been omitted for all entries.

APPENDIX IV

SIMPLE STRUCTURE PLOTS FOR THE FIRST-ORDER ANALYSES

Thurstone[1] set up five useful criteria for determining simple structure. These are as follows:

1. Each row of the oblique factor matrix (Table XV) should have at least one zero.

2. In each column of the oblique factor matrix there should be at least as many zeros as there are factors.

3. For every pair of columns in the oblique factor matrix there should be several tests with zero entries in one column but not in the other.

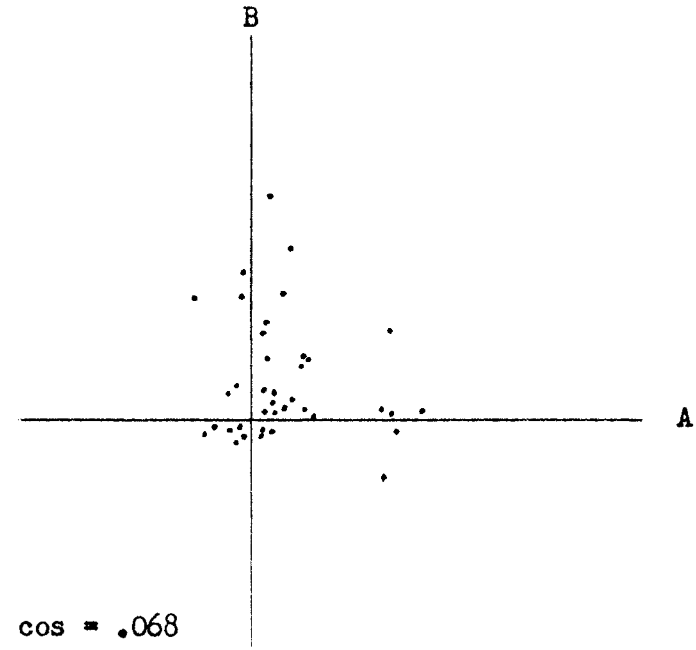
4. For every pair of columns of the oblique factor matrix, a large proportion of the tests should have zero entries in both columns.

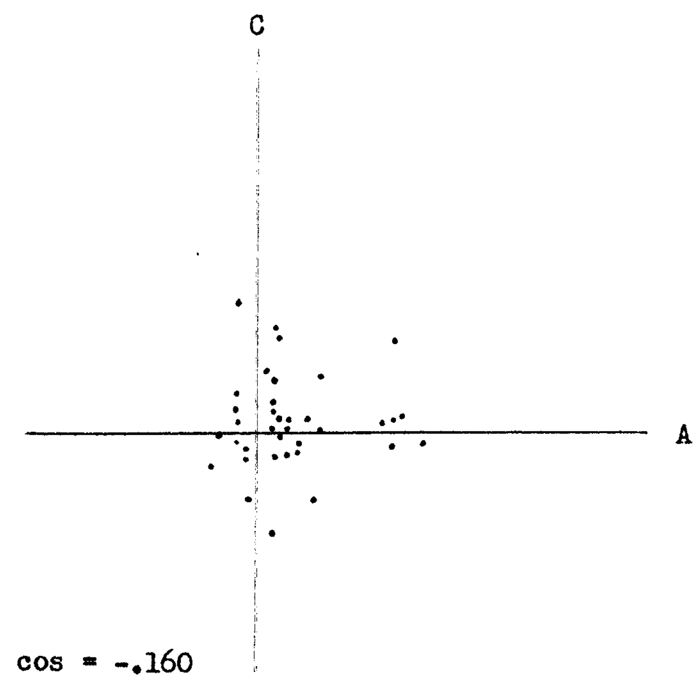
5. For every pair of columns of the oblique factor matrix there should be only a small number of tests with large entries in both columns.

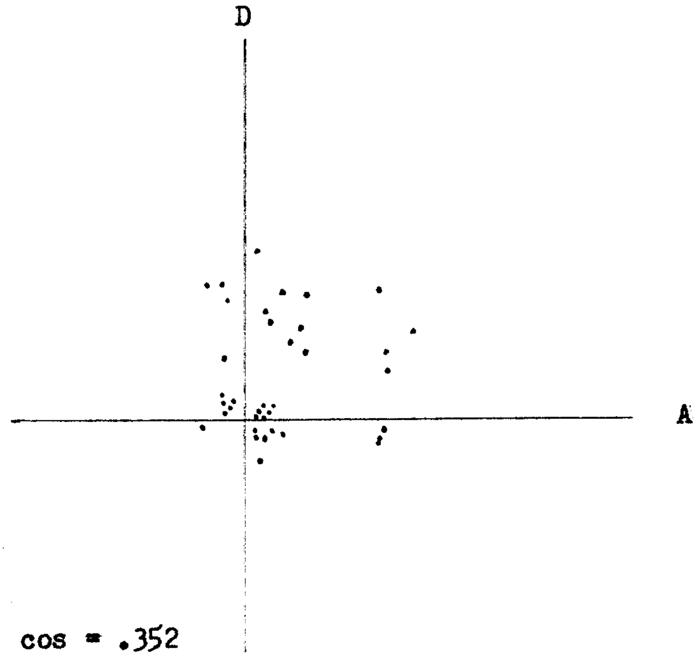
When these conditions are satisfied, the plot of each pair of columns shows (1) a large concentration of points in two radial streaks, (2) a large number of points at or near the origin, and (3) only a small number of points off the two radial streaks.

---

[1] Thurstone, Multiple Factor Analysis, p. 335.


B
A
cos = .068


C
A
cos = -.160



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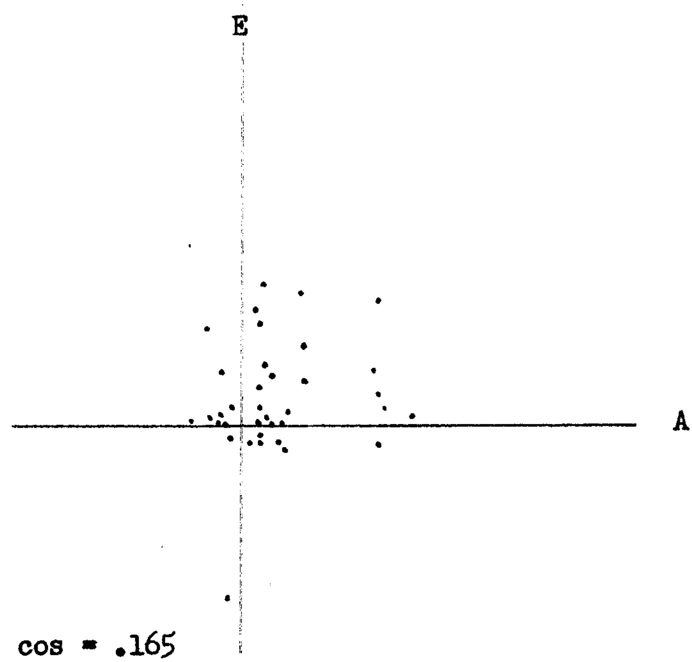

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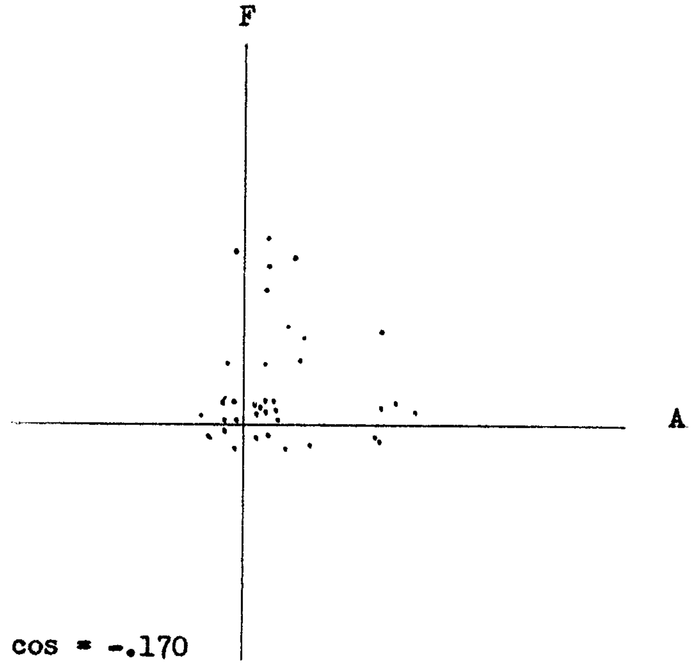

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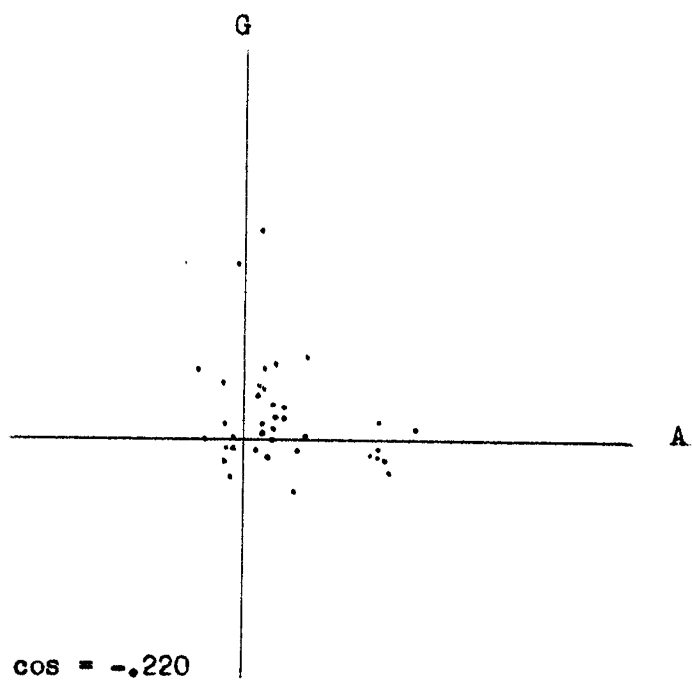

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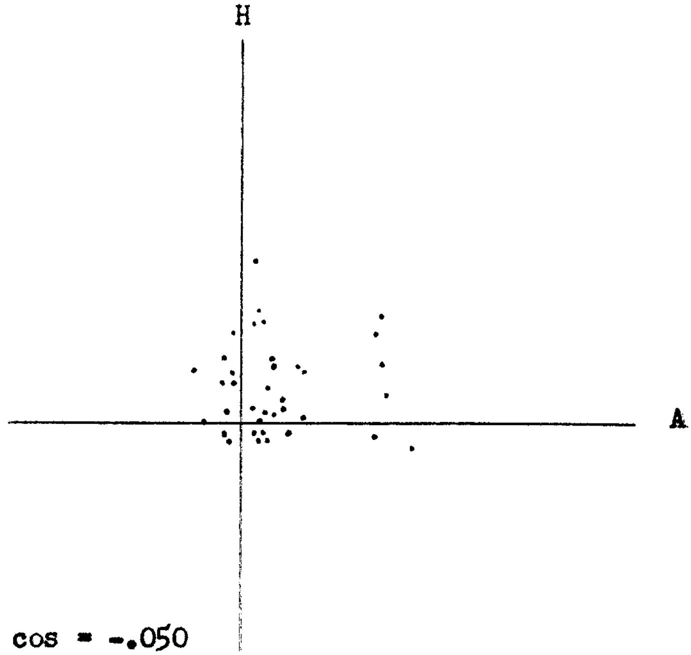

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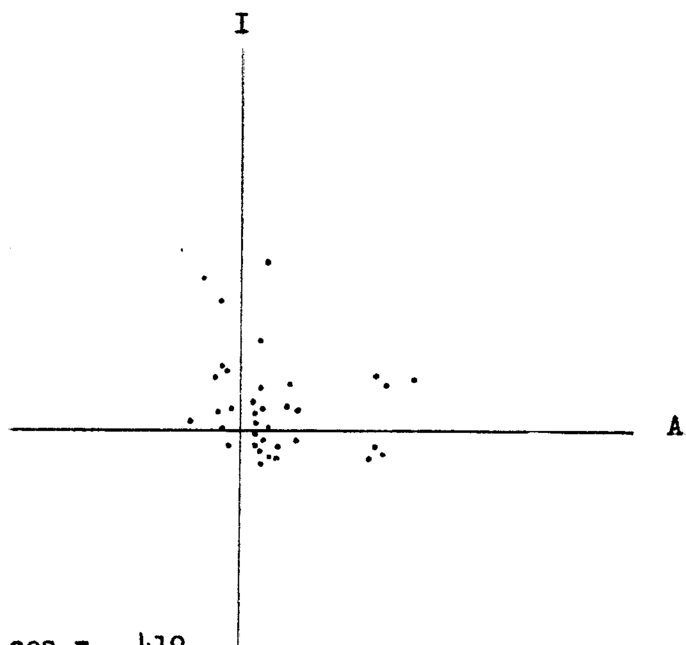

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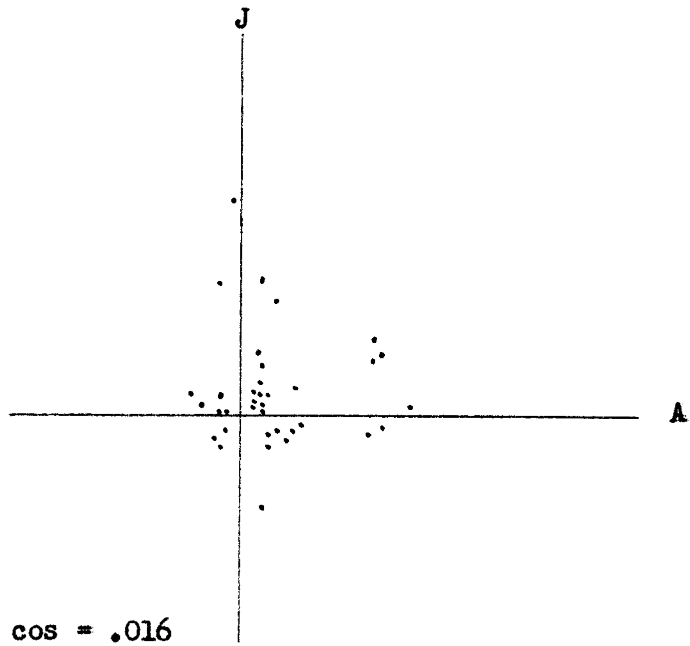

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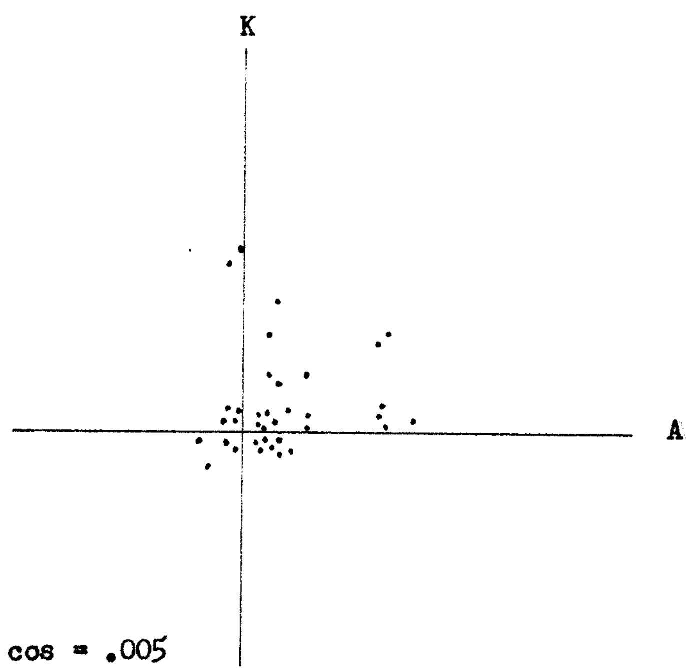

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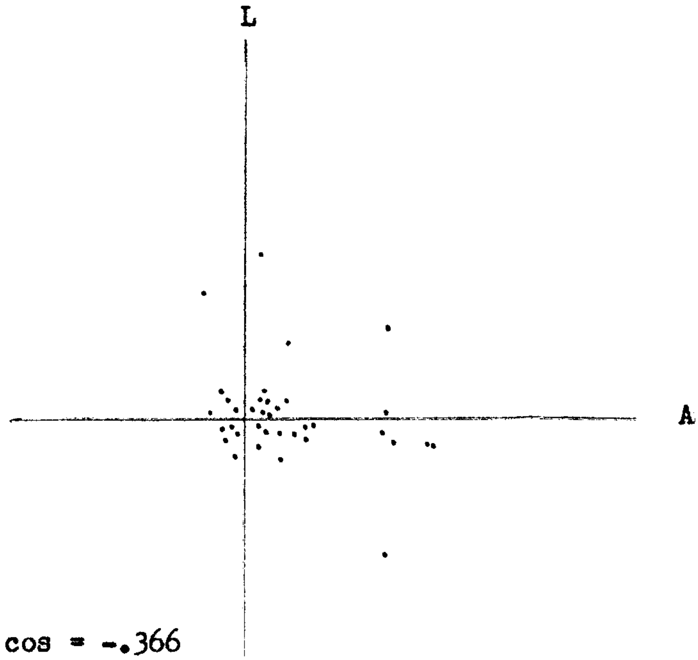

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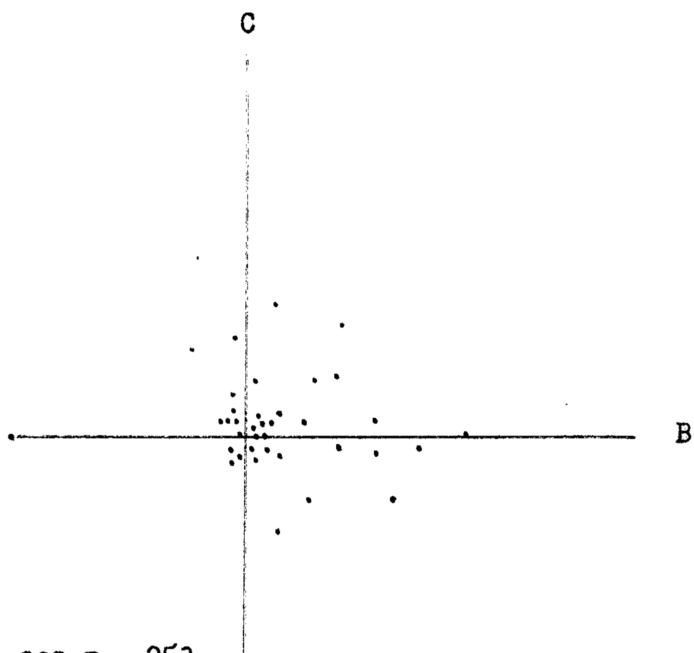

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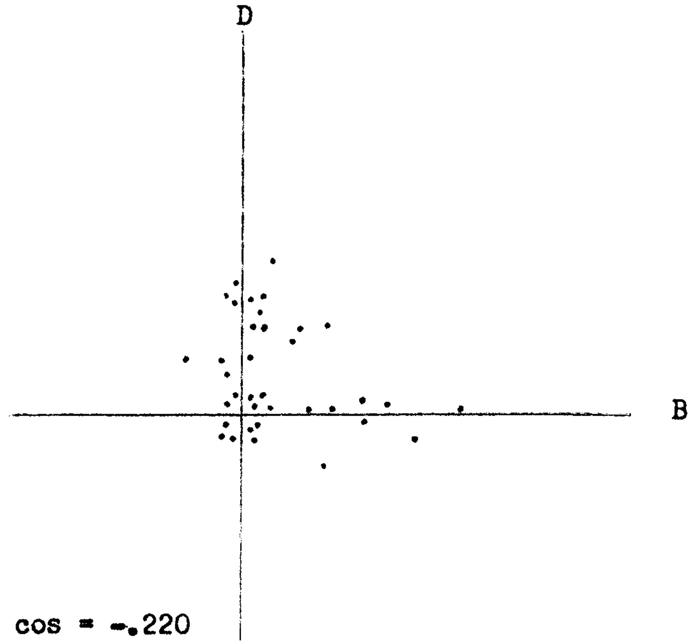

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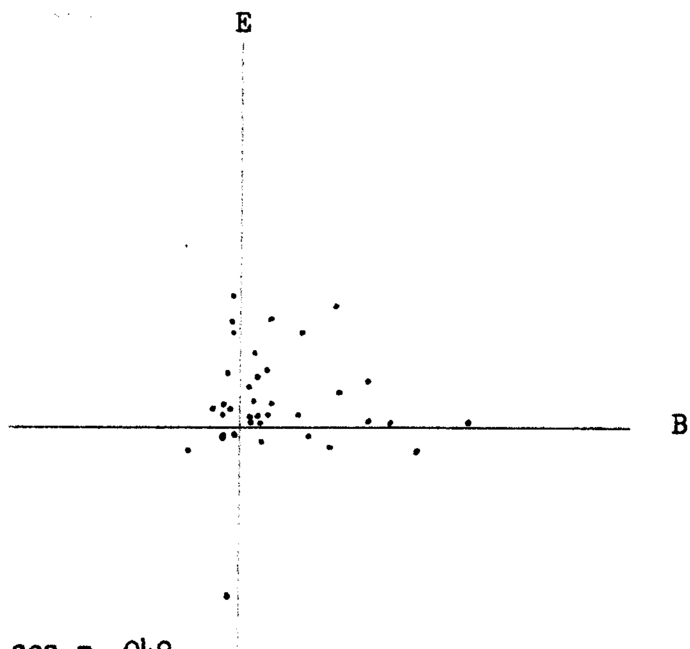

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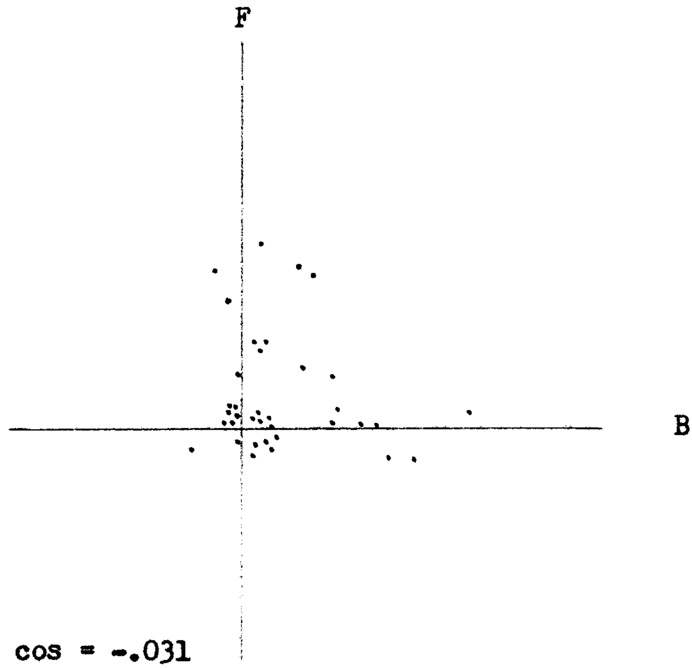

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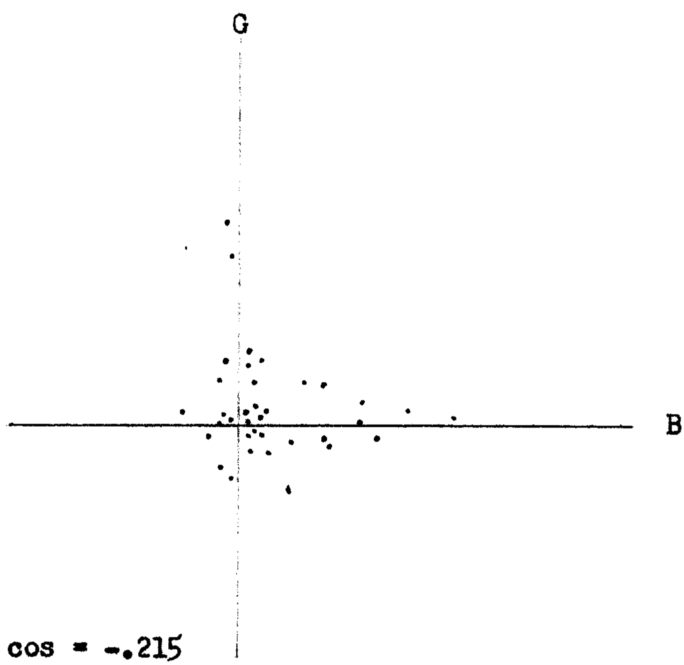

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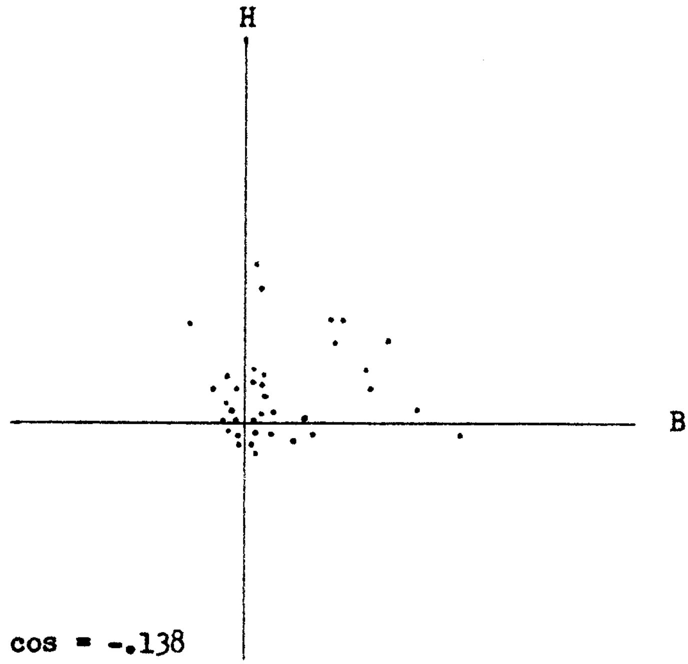

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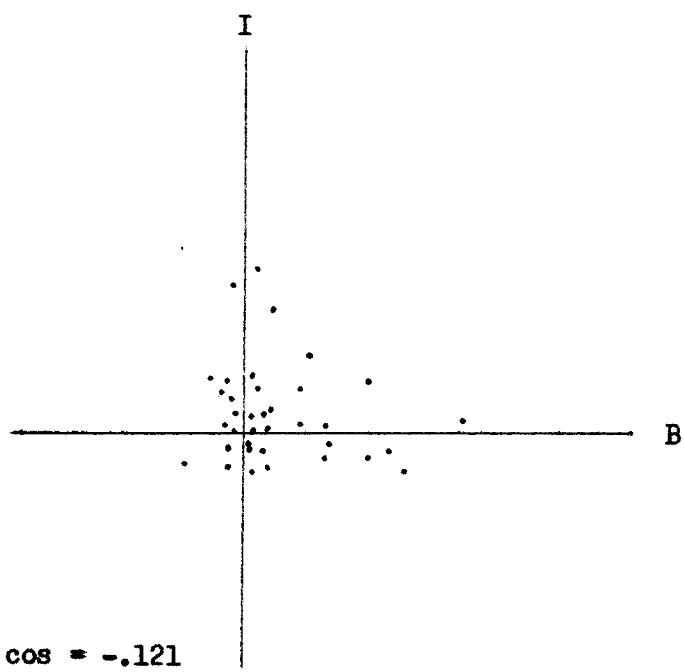

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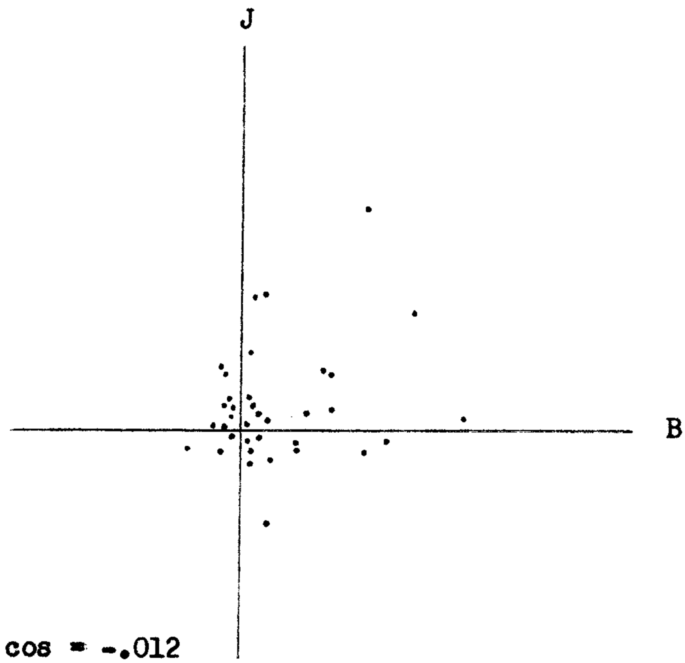

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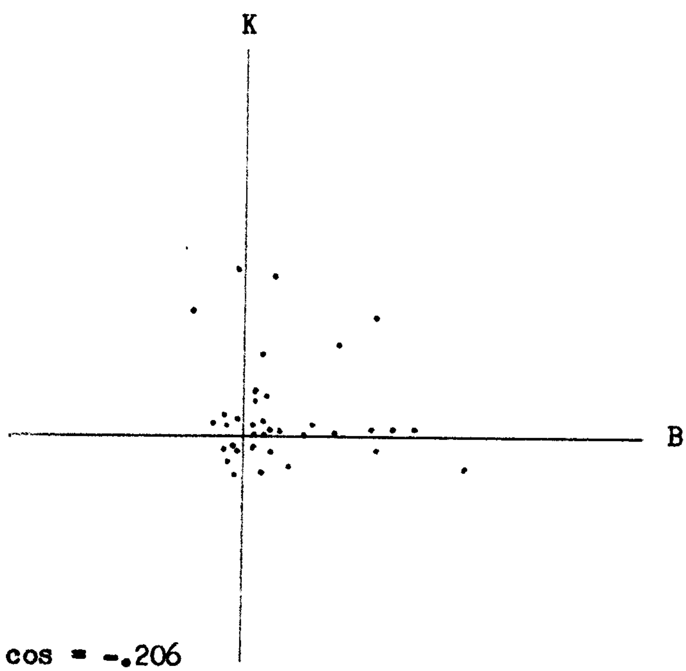

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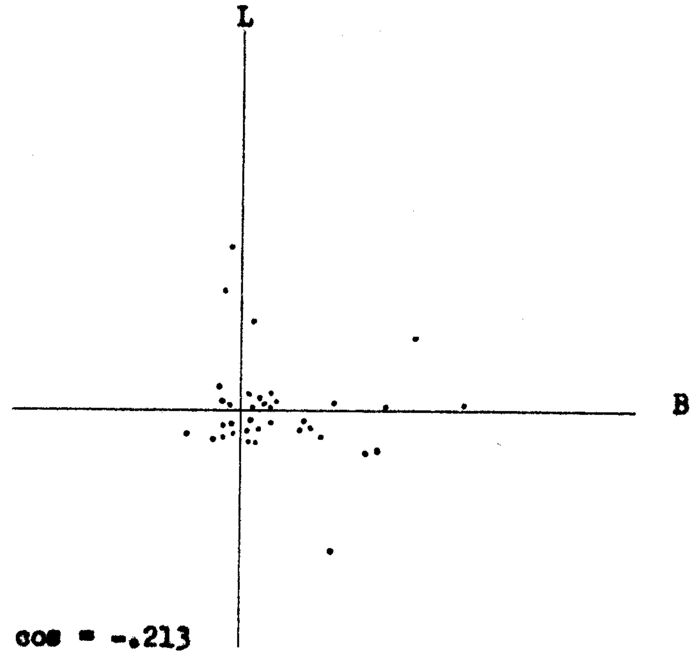
L
B
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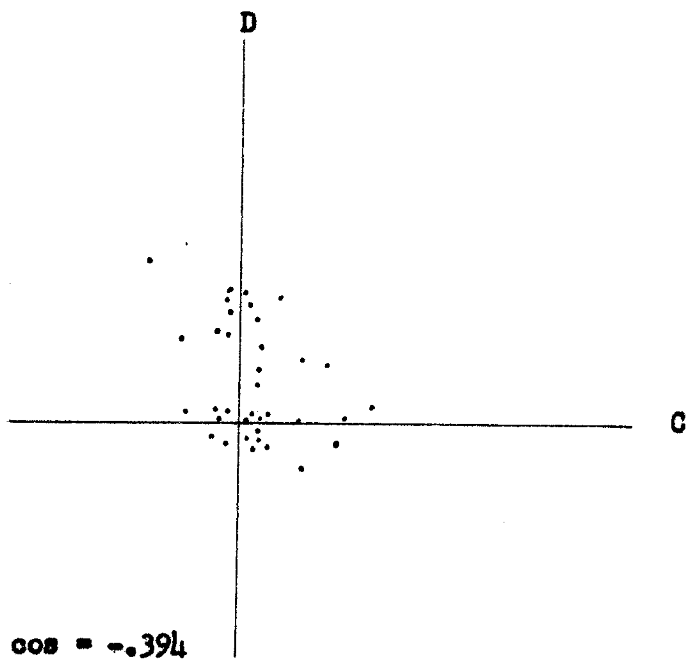
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C
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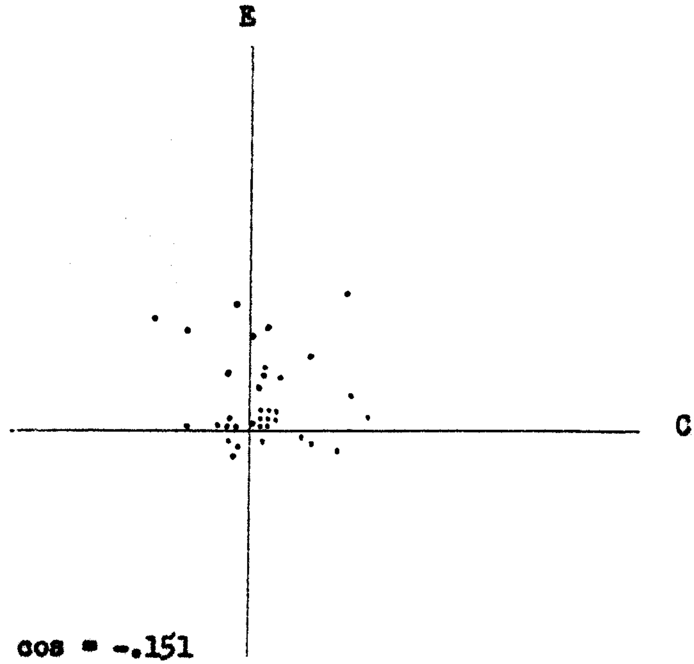

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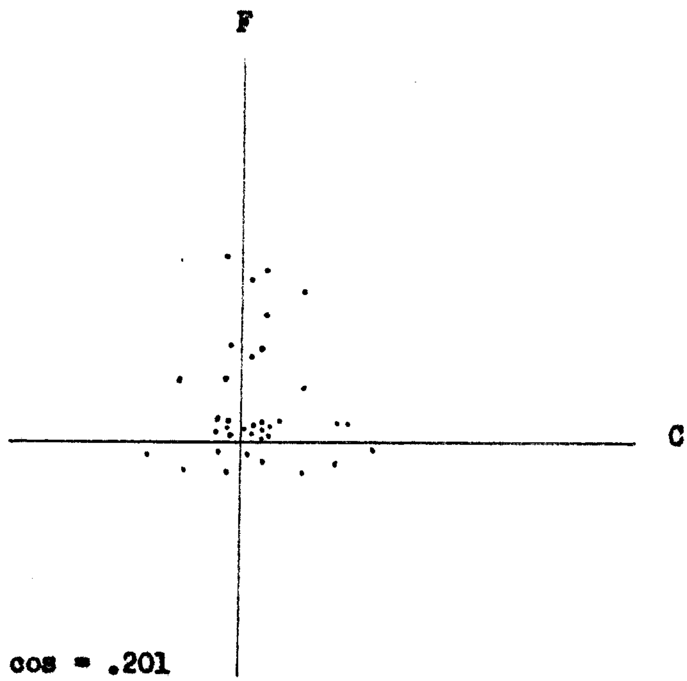

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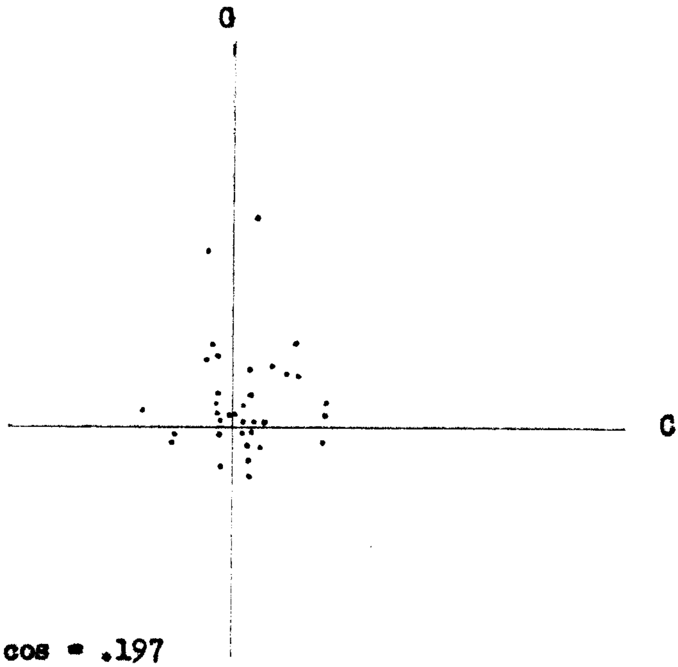

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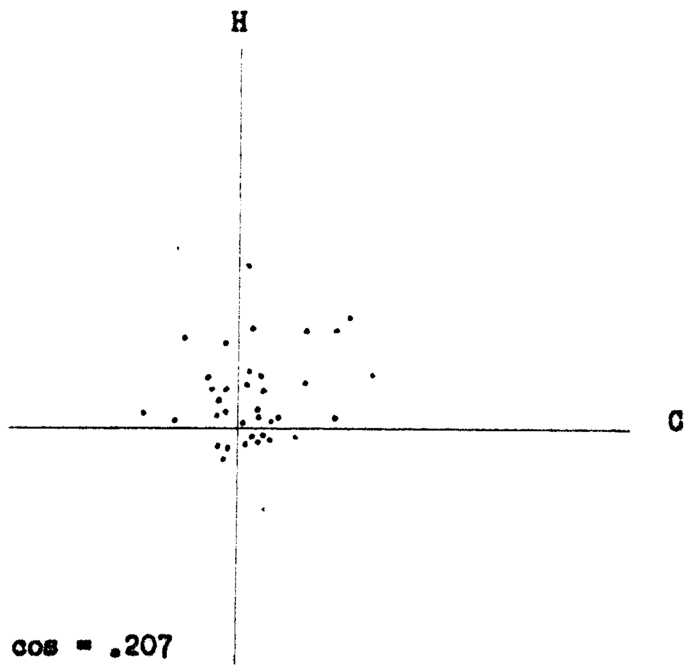

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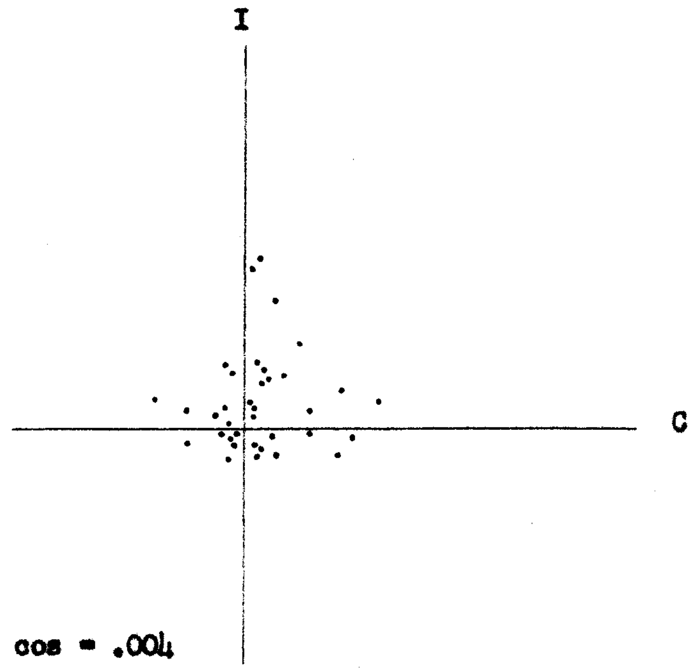
I
C
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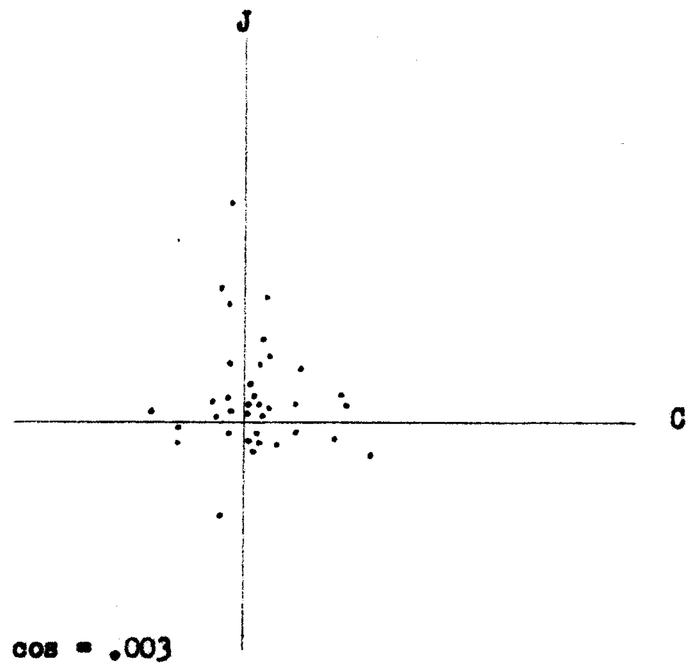
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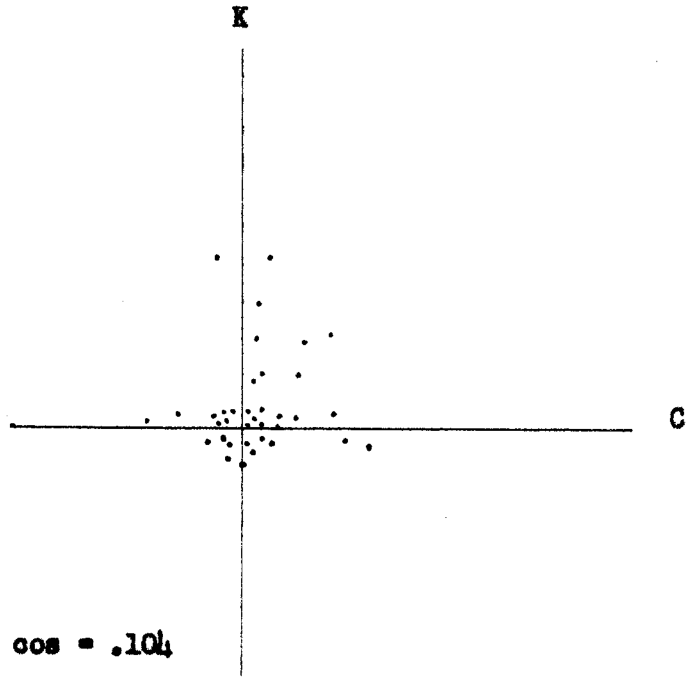

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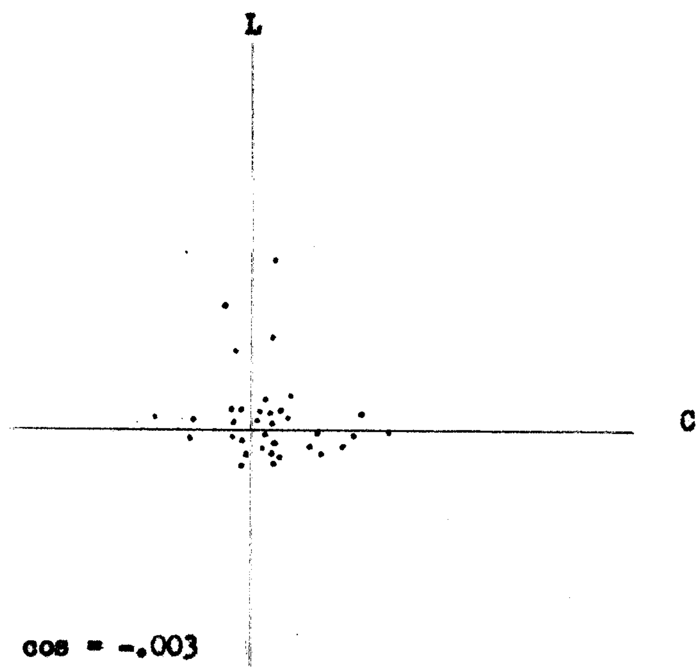

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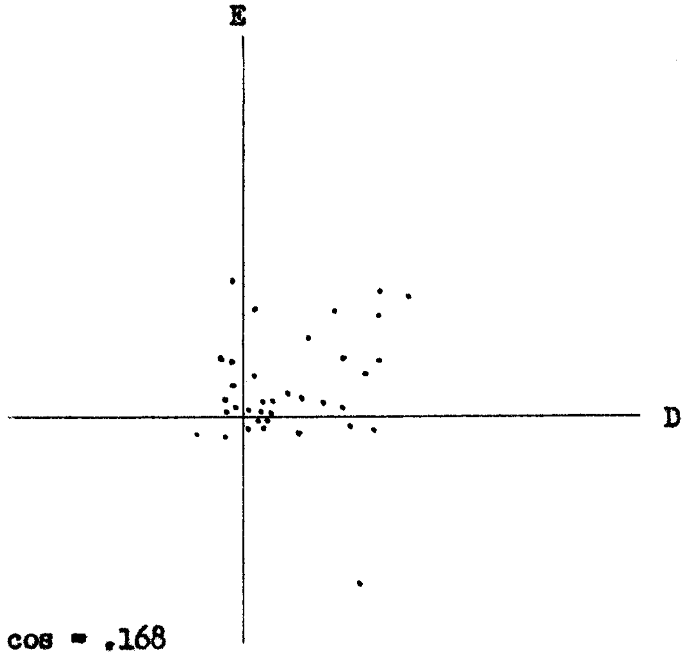

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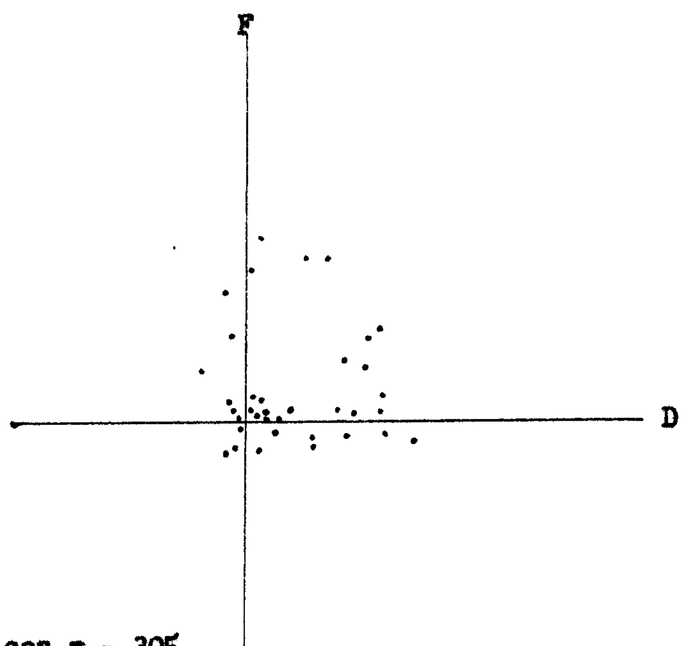

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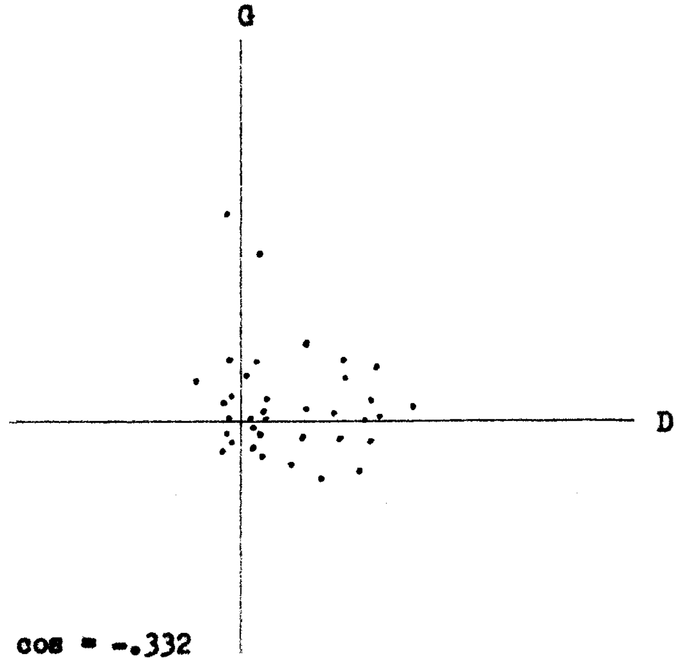
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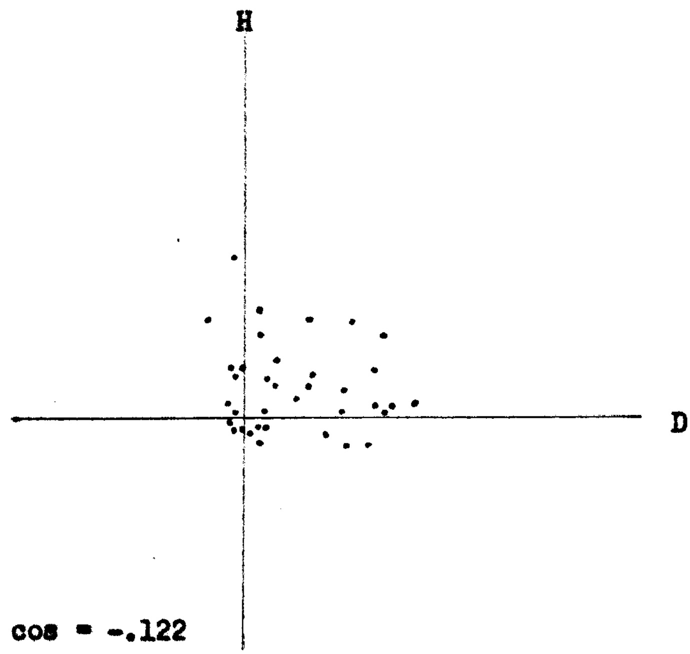
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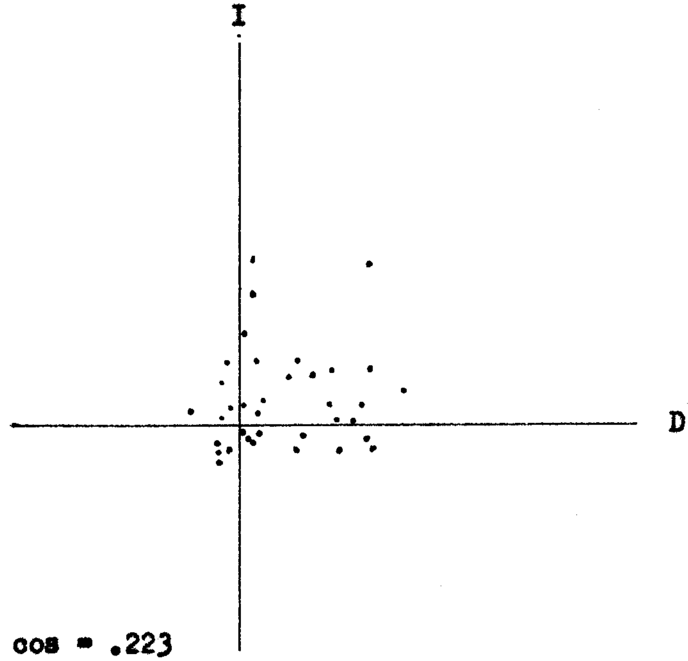

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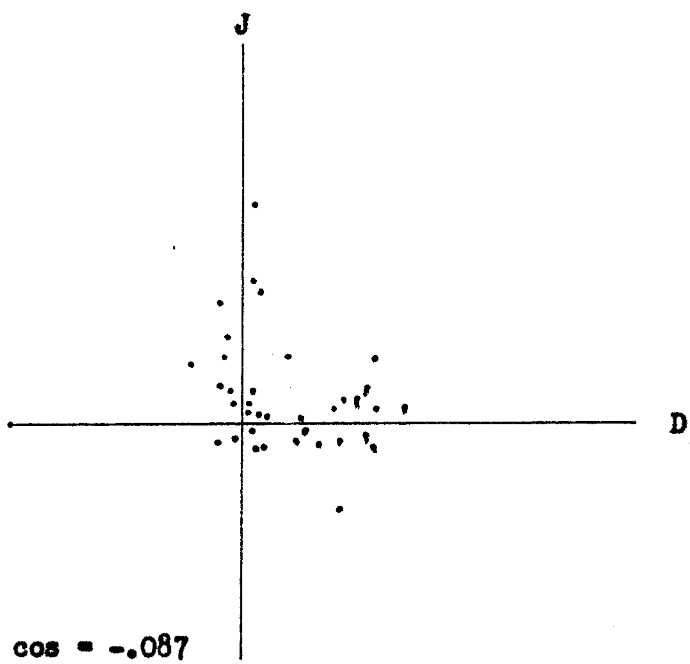

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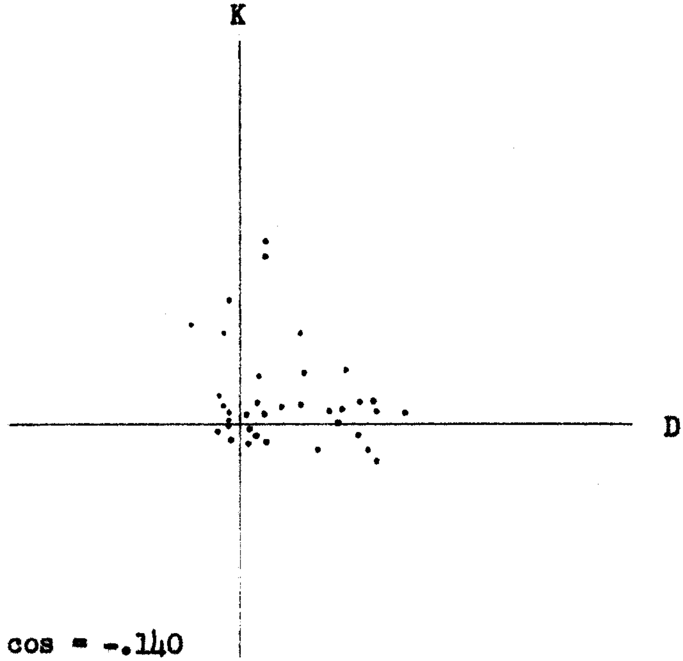

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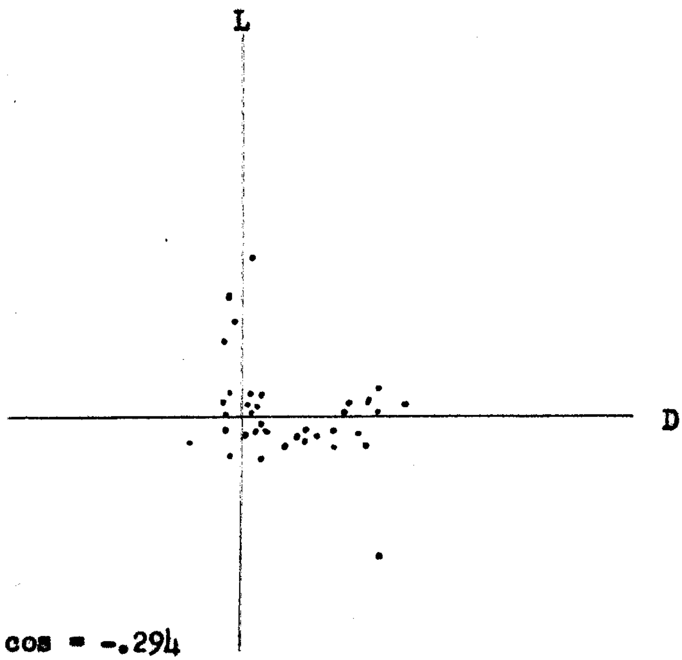

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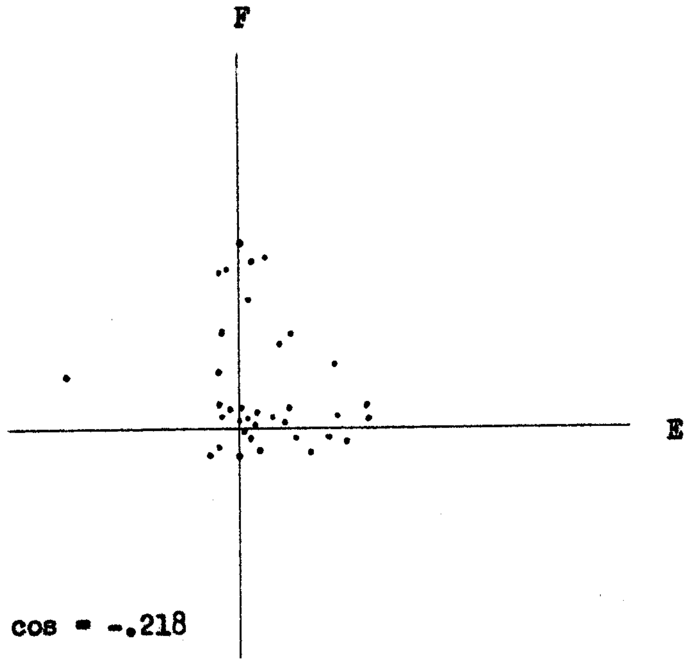
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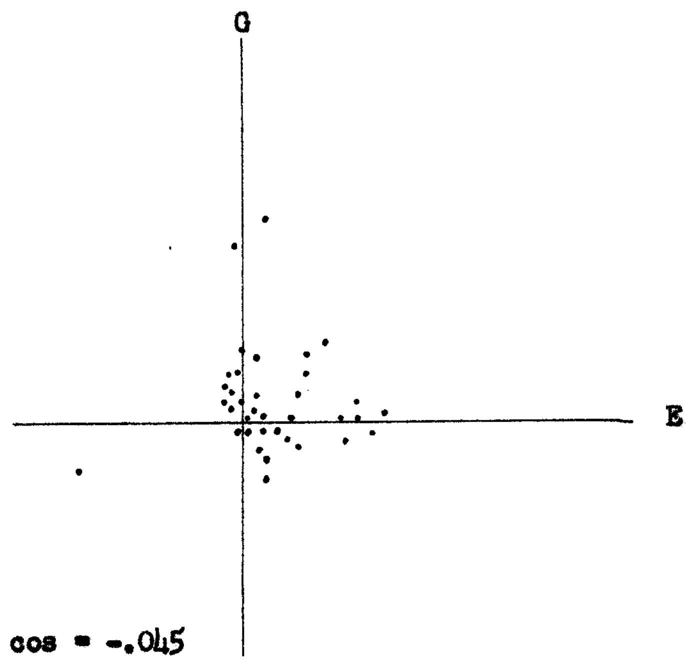
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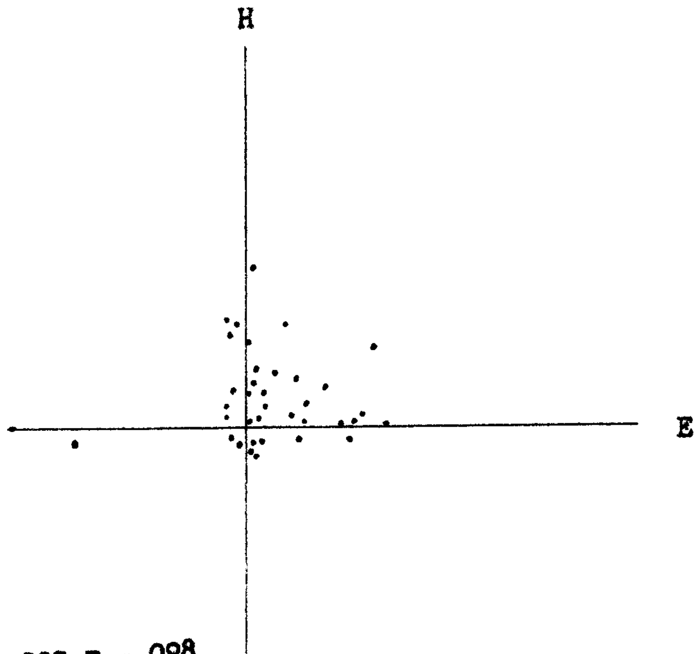

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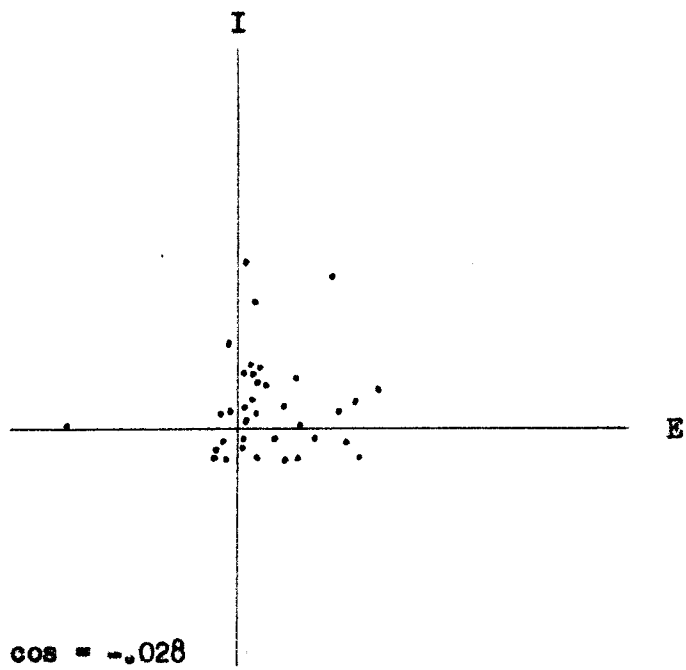

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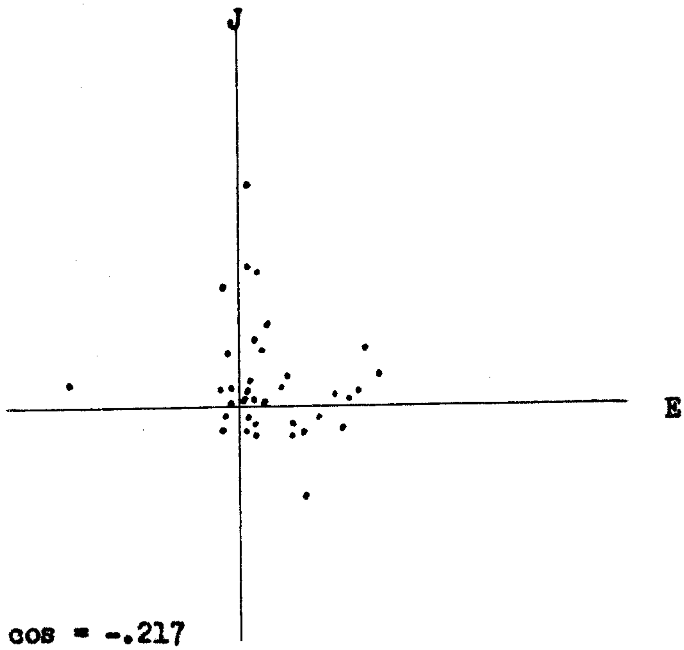

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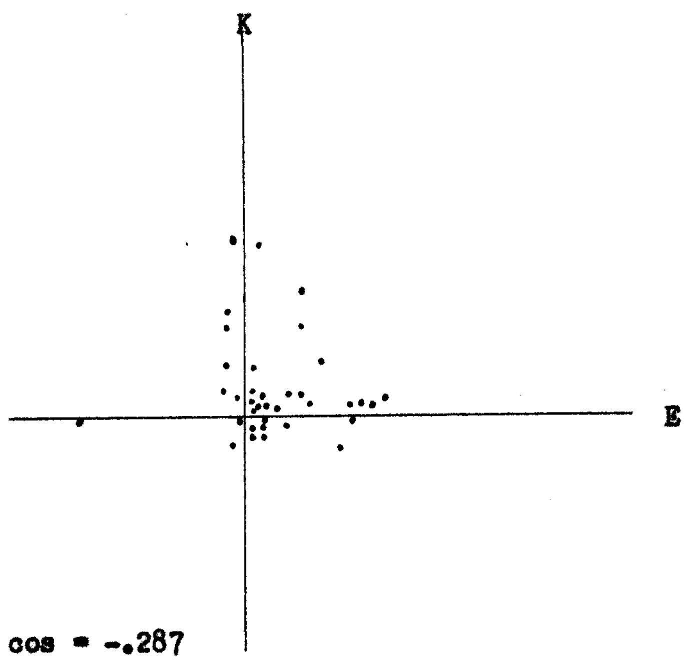

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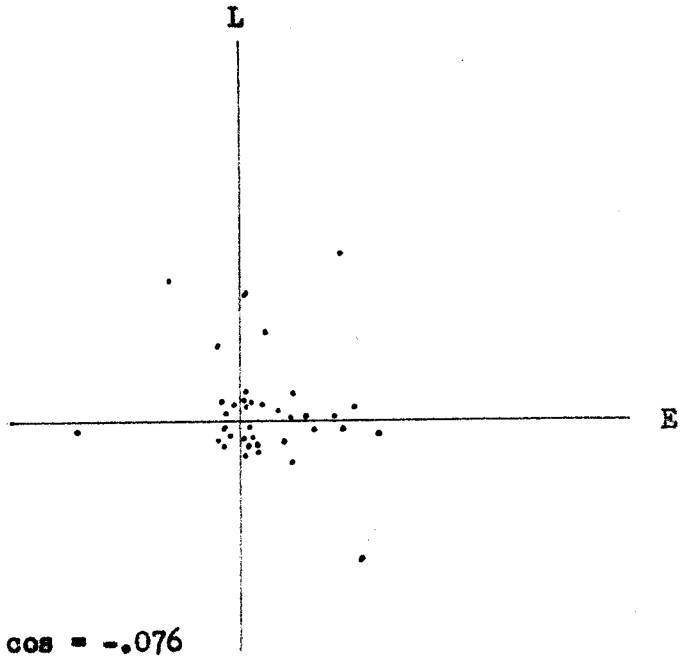

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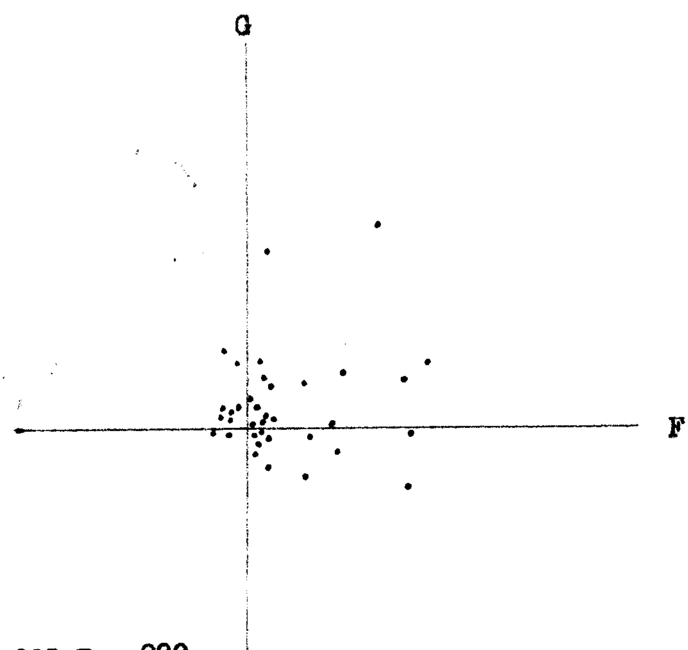

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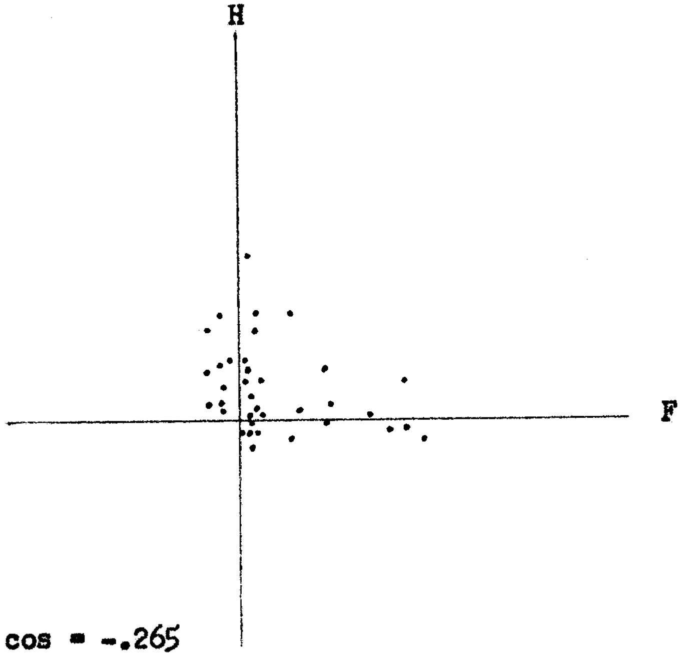

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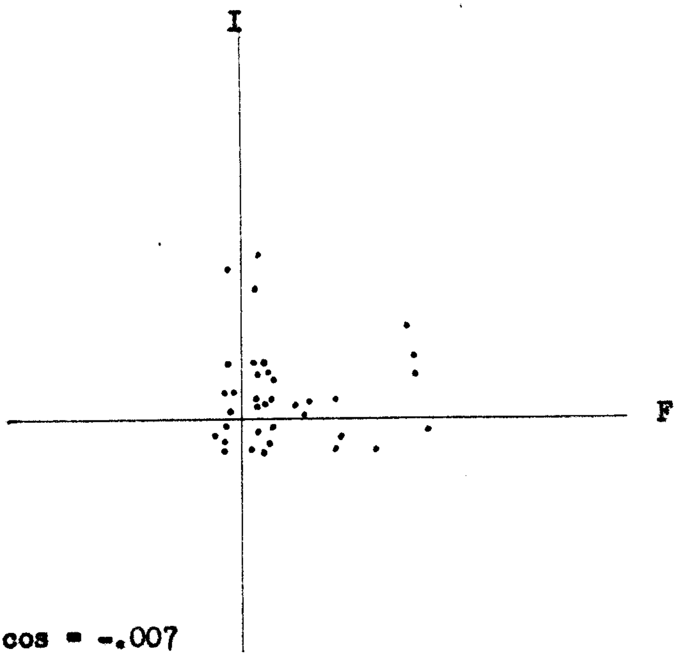

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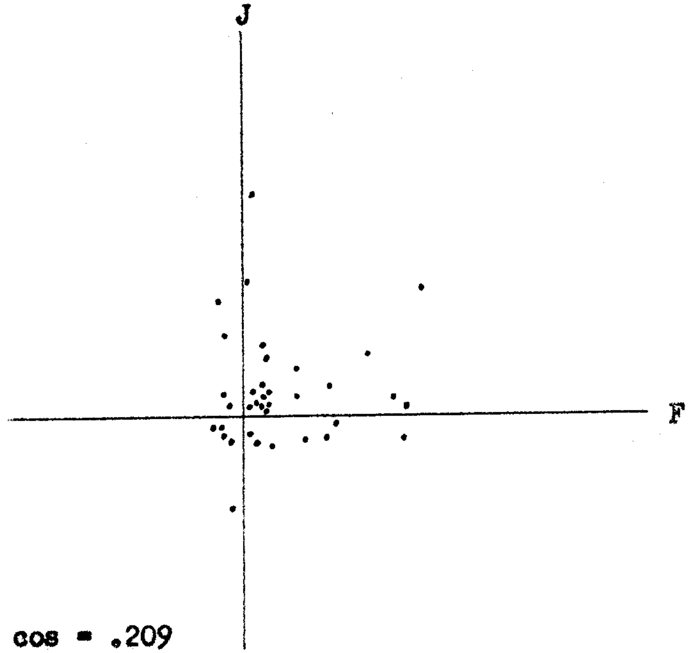

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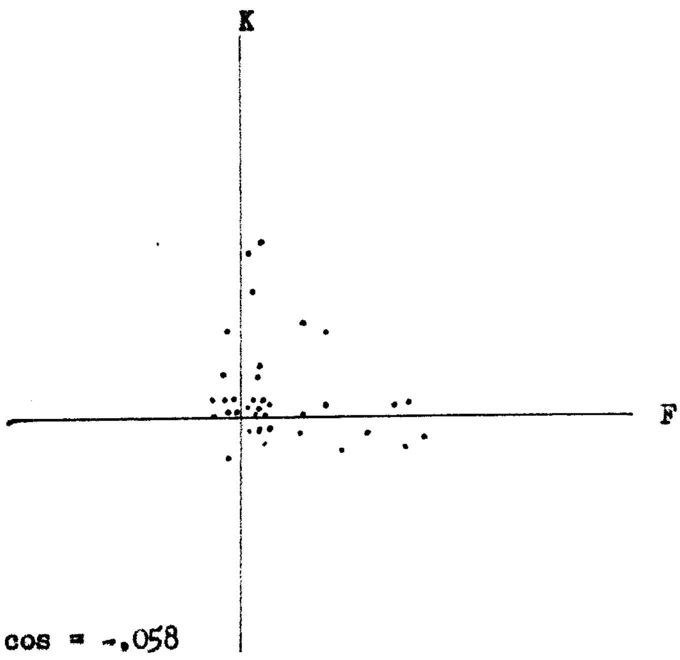

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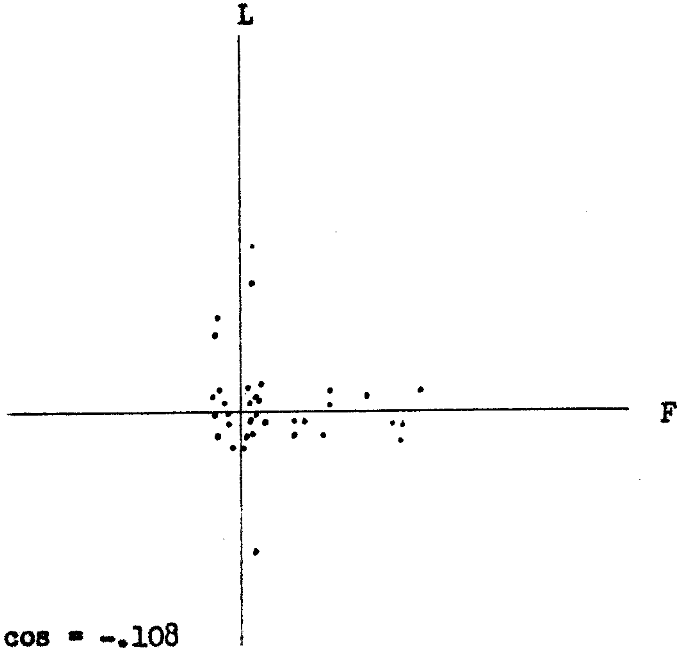

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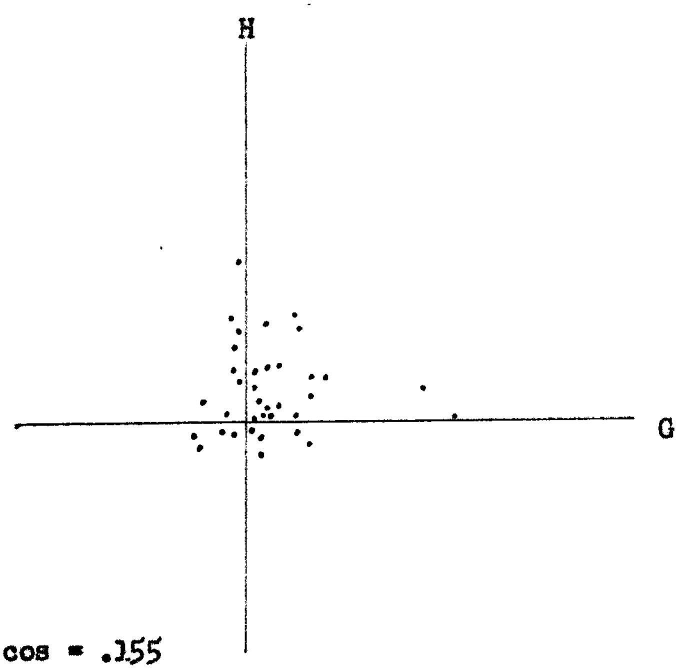

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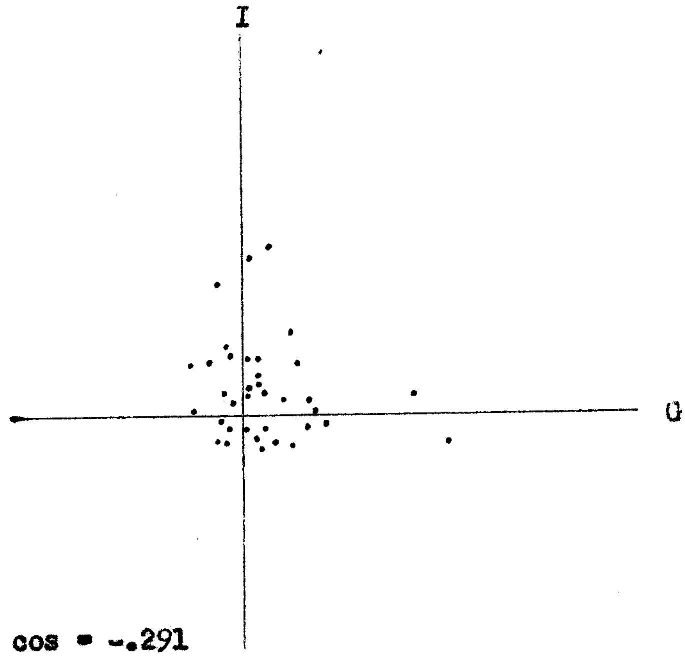

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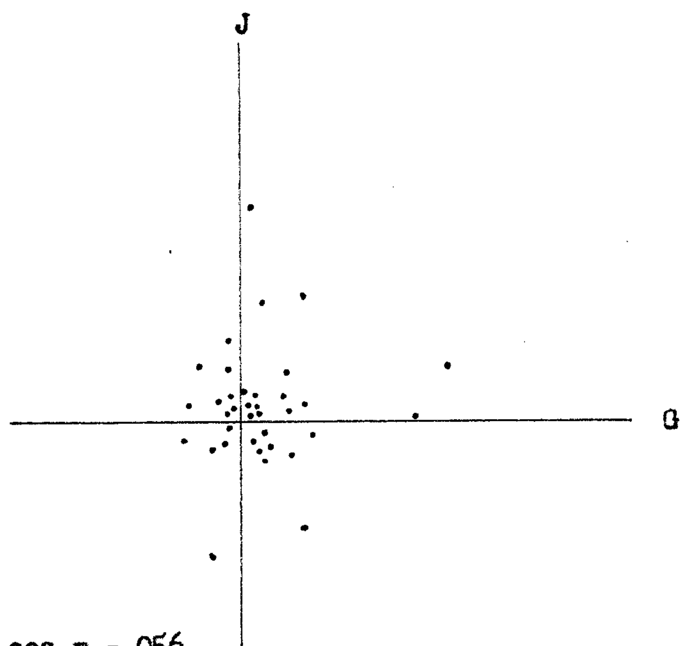

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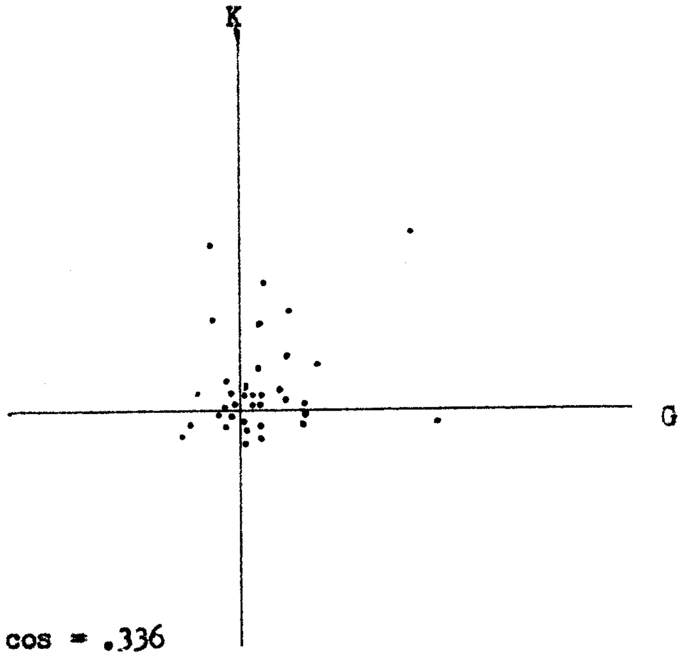

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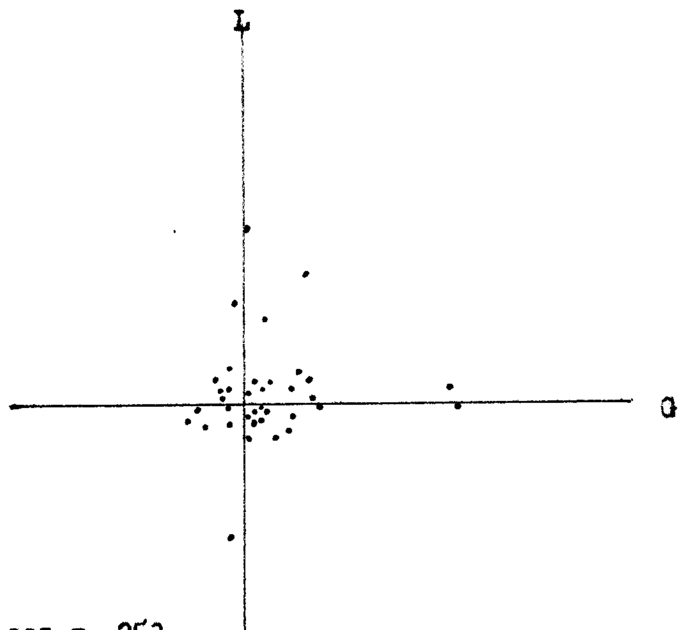

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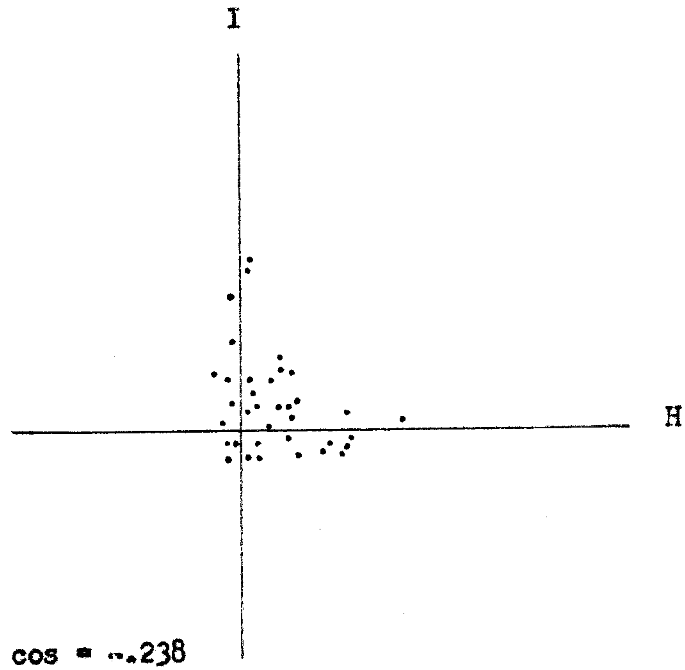

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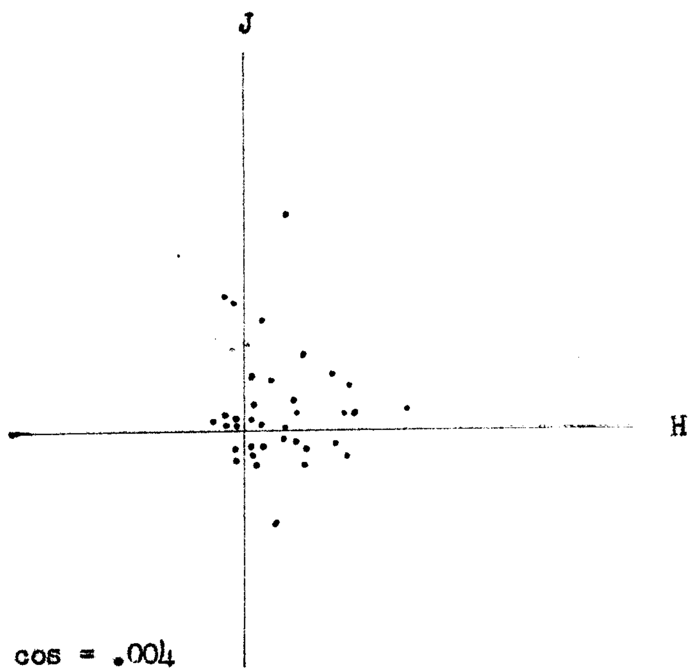

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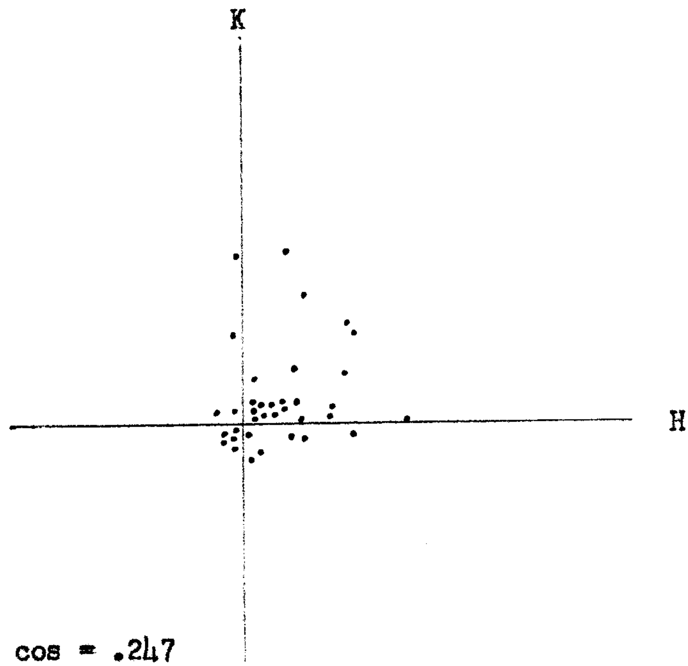

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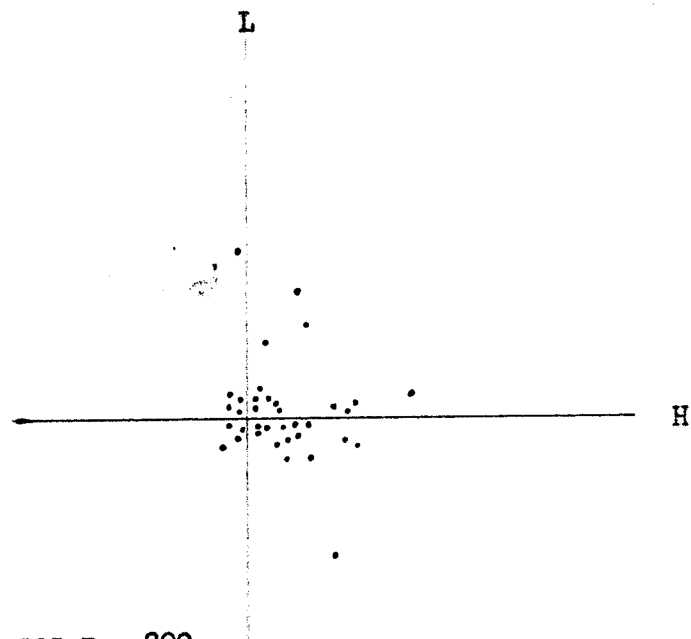

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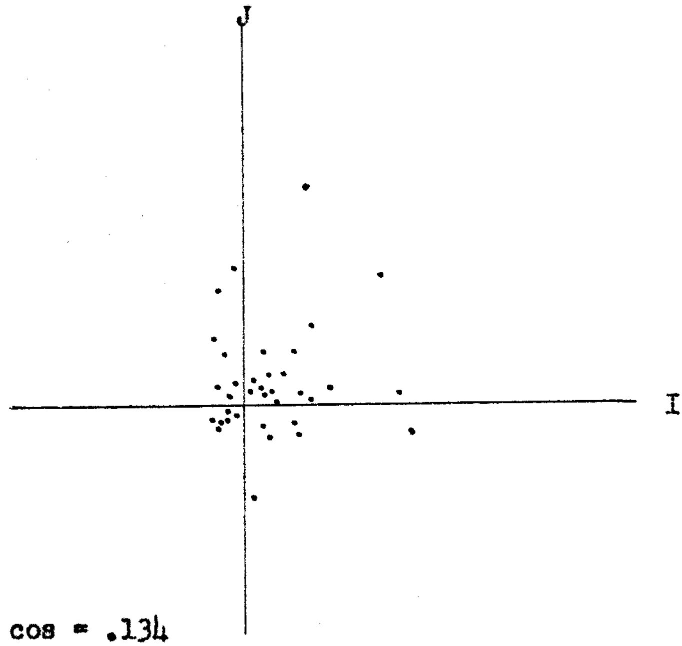

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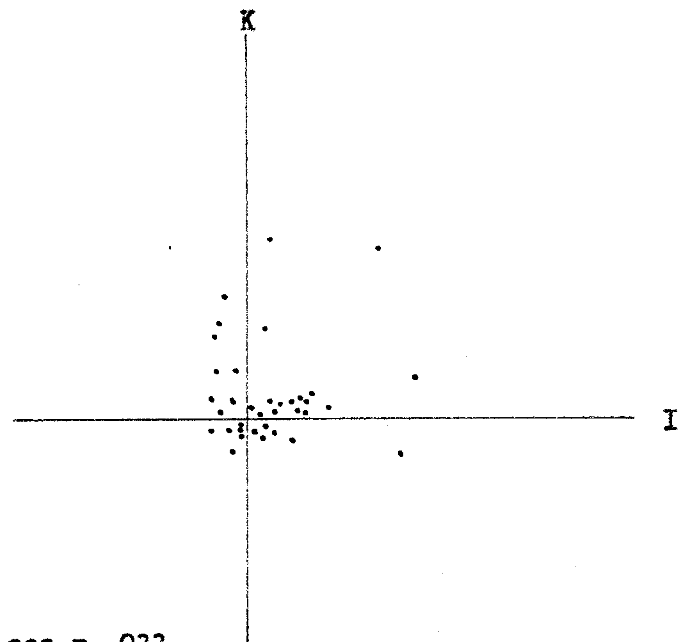

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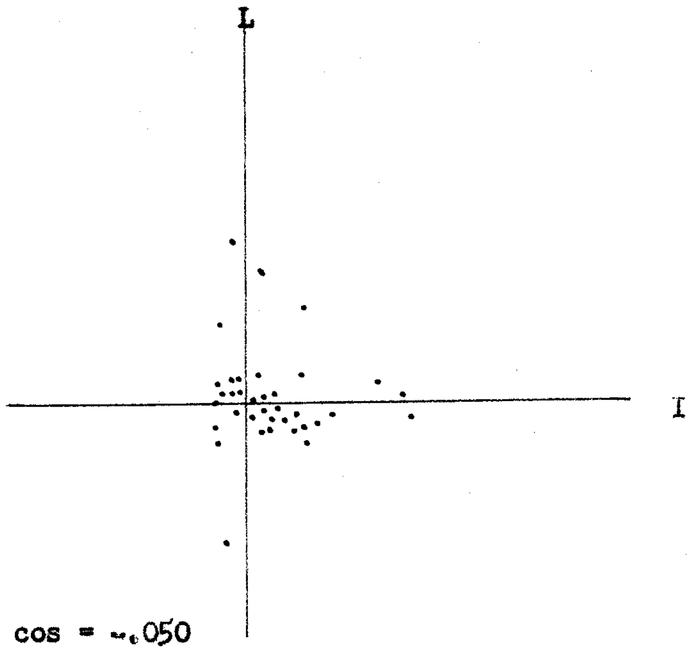

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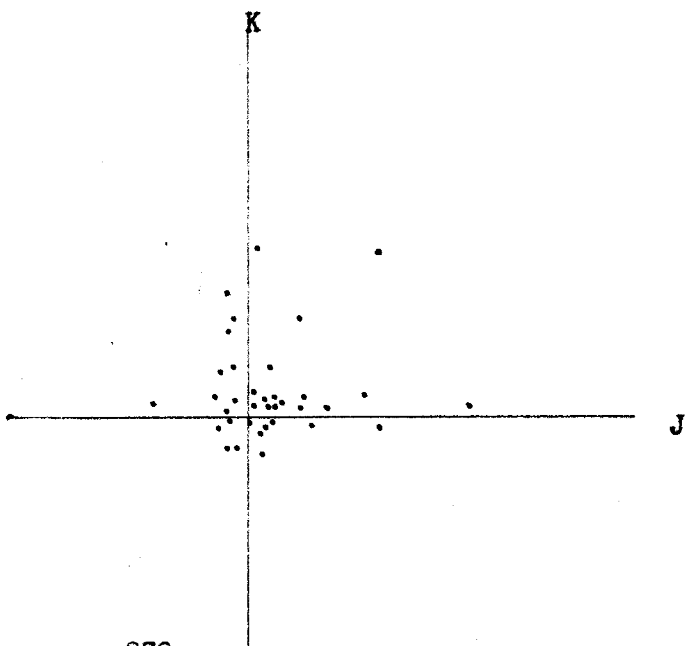

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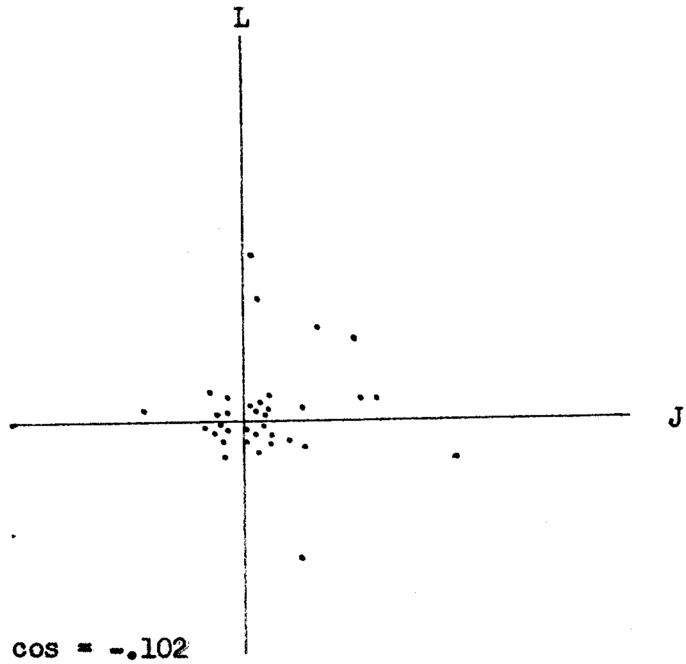
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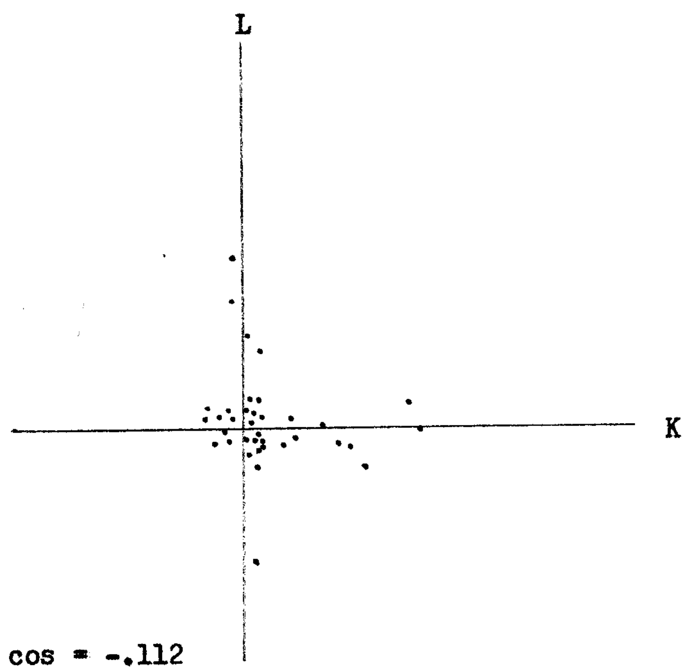
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APPROVAL SHEET

The dissertation submitted by Sister Mary Canisia Majewska, C.S.F.N. has been read and approved by one member of the Department of Psychology and four members of the Department of Education.

The final copies have been examined by the director of the dissertation and the signature which appears below verifies the fact that any necessary changes have been incorporated, and that the dissertation is now given final approval with reference to content, form, and mechanical accuracy.

The dissertation is therefore accepted in partial fulfillment of the requirements for the Degree of Doctor of Philosophy.

Jan 12 1960

Date

Signature of Adviser